

Homogenization of periodic diffusion with small jumps

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Abstract

In this paper, we study the homogenization of a diffusion process with jumps, that is, Feller process generated by an integro-differential operator. This problem is closely related to the problem of homogenization of boundary value problems arising in studying the behavior of heterogeneous media. Under the assumptions that the corresponding generator has vanishing drift coefficient, rapidly periodically oscillating diffusion and jump coefficients, that it admits only “small jumps” (that is, the jump kernel has finite second moment) and under certain additional regularity conditions, we prove that the homogenized process is a Brownian motion. The presented results generalize the classical and well-known results related to the homogenization of a diffusion process.

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1 Introduction

Many phenomena arising in nature, engineering and social sciences involve heterogeneous media, such as problems related to diffusion of population, composite materials and large financial market movements. Because of heterogeneity, mathematical models used in describing these phenomena (typically stochastic processes or integro-differential equations) are characterized by heterogeneous coefficients, and as such are very complicated and hard to analyze. However, on macroscopic scales, they often show an effective behavior which is a result of macroscopic scale averaging of the complicated microscopic scale structure. More precisely, in many cases when the coefficients (rapidly) vary on small scales it is possible to use the fine microscopic structure of the media to derive an effective (homogenized) model which is a valid approximation of the initial model and, in general, it is of much simpler form (typically it is characterized by constant coefficients).

The goal of this paper is to investigate the homogenization of a d -dimensional Feller process $\{F_t^\varepsilon\}_{t \geq 0}$ generated by a non-local (integro-differential) operator (with vanishing drift term) of the

form

$$\mathcal{A}_\varepsilon f(x) := \frac{1}{2} \sum_{i,j=1}^d c_{ij}(x/\varepsilon) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \left(f(\varepsilon y + x) - f(x) - \varepsilon \sum_{i=1}^d y_i \frac{\partial f(x)}{\partial x_i} 1_{B(0,1)}(y) \right) \nu(x/\varepsilon, dy), \quad (1.1)$$

the so-called (zero-drift) diffusion process with jumps (see Section 2 for details). Here, as usual, $\varepsilon > 0$ is a small parameter defined as a microstructure period intended to tend to zero. We assume that the coefficients $c(x) = (c_{ij}(x))_{1 \leq i,j \leq d}$ and $\nu(x, dy)$ are periodic (say with period $\tau > 0$ in all coordinate directions), $c(x)$ is symmetric and non-negative definite and $\nu(x, dy)$ is a Borel kernel satisfying

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |y|^2 \nu(x, dy) < \infty. \quad (1.2)$$

As we have commented, due to the rapidly oscillating nature of the coefficients of \mathcal{A}_ε , it is expected that the microscopic behavior (heterogeneity) can be averaged out to result a simpler model approximating the behavior at the macroscopic level (the homogenization occurs). Indeed, as the main result of this paper, we prove that as $\varepsilon \searrow 0$, in the path space endowed with the Skorokhod topology, the homogenized process is a zero-drift Brownian motion (or, equivalently, the homogenized operator is a zero-drift second-order elliptic operator) determined by a covariance matrix (diffusion coefficient) of the form

$$\left(\int_{[0,\tau]} c_{ij}(x) \pi(dx) + \int_{[0,\tau]} \int_{\mathbb{R}^d} y_i y_j \nu(x, dy) \pi(dx) \right)_{1 \leq i,j \leq d},$$

where $\pi(dx)$ is an invariant probability measure associated to the projection of $\{F_t^1\}_{t \geq 0}$ on the d -dimensional torus $[0, \tau]^d$ (see Section 4 for details). Note that, in view of the assumption in (1.2), it is very natural to choose the diffusive scaling in (1.1). Namely, $\{F_t^\varepsilon\}_{t \geq 0}$ can be seen as $\{\varepsilon F_{\varepsilon^{-2}t}^1\}_{t \geq 0}$ (both processes are generated by \mathcal{A}_ε) which naturally leads to a diffusive limit.

Further, observe that the process $\{F_t^\varepsilon\}_{t \geq 0}$ (operator \mathcal{A}_ε) can be seen as a classical (zero-drift) diffusion process (second-order elliptic operator) perturbed by a jump process (non-local operator) which, in many situations, makes the whole model more realistic. For example, such processes (operators) are used to model population dynamics also by taking into account long-range effects (see [CF05] and the references therein). In these models the kernel $\nu(x, dy)$ is typically assumed to be a probability kernel representing a non-local dispersal law of the population (for example, induced by visual contact or chemical reaction). Also, such models are nowadays a standard tool in representing the risk related to large market movements (see [CT04]). Namely, due to the shortcomings of diffusion models in modeling these type of phenomena, various models (processes) with jumps of type (1.1) were introduced, which allow for more realistic representation of price dynamics and a greater flexibility in modeling. Observe that, from the practical point of view, in both contexts it is very natural to assume that the kernel $\nu(x, dy)$ has compact support or, equivalently, the corresponding diffusion with jumps has uniformly bounded jumps. Hence, the condition in (1.2) is automatically satisfied. As we have already commented, the (small) parameter $\varepsilon > 0$ characterizes the size of the cell of periodicity of the media. For example, in the problem of forest migration (see [PZ04]) it represents the area of actively dispersed seeds (by animal cachers). More precisely, the process of active seed dispersion occurs on scales much smaller than forest migration and, clearly, it is natural to assume that the structure of the environment (cache areas) is more or less periodic. Also, note that the non-local effect in the model is reflected through disturbances like wind, fire and avian dispersers.

The work in this paper is highly motivated by the works in [BLP78] and [Bha85] in which the authors have considered the homogenization of a diffusion $\{F_t^\varepsilon\}_{t \geq 0}$ generated by second-order elliptic operator of the form

$$\mathcal{A}_\varepsilon f(x) := \frac{1}{\varepsilon} \sum_{i=1}^d b_i(x/\varepsilon) \frac{\partial f(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d c_{ij}(x/\varepsilon) \frac{\partial^2 f(x)}{\partial x_i \partial x_j}. \quad (1.3)$$

Under the assumptions that the coefficients $b(x) = (b_i(x))_{1 \leq i \leq d}$ and $c(x) = (c_{ij}(x))_{1 \leq i,j \leq d}$ are periodic and smooth enough (see Section 6 for details) and $c(x)$ is symmetric and uniformly elliptic, they have shown that the homogenized process (operator) of the diffusion $\{F_t^\varepsilon - \varepsilon^{-1} \bar{b}t\}_{t \geq 0}$ (generator $\mathcal{A}_\varepsilon f(x) - \frac{1}{\varepsilon} \sum_{i=1}^d \bar{b}_i \frac{\partial f(x)}{\partial x_i}$) is a zero-drift Brownian motion (zero-drift second-order elliptic operator) determined by a covariance matrix (diffusion coefficient) of the form

$$\bar{\Sigma} := \left(\int_{[0,\tau]^d} \sum_{k,l=1}^d \left(\delta_{ki} - \frac{\partial \beta_i(x)}{\partial x_k} \right) c_{kl}(x) \left(\delta_{lj} - \frac{\partial \beta_j(x)}{\partial x_l} \right) \pi(dx) \right)_{1 \leq i,j \leq d}. \quad (1.4)$$

Here, $\bar{b} := (\bar{b}_i)_{1 \leq i \leq d}$ for $\bar{b}_i := \int_{[0,\tau]} b_i(x) \pi(dx)$, $\pi(dx)$ is the unique invariant probability measure corresponding to the projection of $\{F_t^1\}_{t \geq 0}$ on the d -dimensional torus $[0, \tau]^d$ (see Section 4 for details), and $\beta_i \in C^2(\mathbb{R}^d)$, $i = 1, \dots, d$, are unique τ -periodic solutions to $\mathcal{A}_1 \beta_i(x) = b_i(x) - \bar{b}_i$ (see Section 6 for details). For more on the homogenization of local operators (methods and applications) we refer the readers to the monographs [All02], [BLP78], [JKO94], [CPS07] and [Tar09].

In the closely related paper [RV09] the authors have considered the homogenization of a one-dimensional diffusion with jumps in ergodic random environment $(\Omega, \mathcal{F}, \mathbb{P}, \{\tau_x\}_{x \in \mathbb{R}})$, generated by an operator of the form (in a fixed random environment $\omega \in \Omega$)

$$\begin{aligned} \mathcal{A}_\varepsilon^\omega f(x) &:= \frac{1}{\varepsilon} b(\tau_{x/\varepsilon} \omega) f'(x) + \frac{1}{2} c(\tau_{x/\varepsilon} \omega) f''(x) \\ &+ \frac{1}{\varepsilon^2} \int_{\mathbb{R}} (f(\varepsilon y + x) - f(x) - \varepsilon y f'(x) 1_{B(0,1)}(y)) \nu(\tau_{x/\varepsilon} \omega, dy), \end{aligned} \quad (1.5)$$

where $\{\tau_x\}_{x \in \mathbb{R}}$ is an ergodic group of measure preserving transformations acting on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By assuming that $\int_{\mathbb{R}} |y|^2 \nu(\omega, dy) < \infty$ for all $\omega \in \Omega$ (and certain regularity conditions on the coefficients of $\mathcal{A}_\varepsilon^\omega$), they have shown that the homogenized process is a Brownian motion. Observe that if we remove the environment in (1.5), the underlying process becomes a Lévy process (diffusion with jumps with constant coefficients) whose homogenization we discuss in Corollary 5.2. Let us remark that the authors have also discussed the case when $\int_{\mathbb{R}} |y|^2 \nu(\omega, dy) = \infty$ for all $\omega \in \Omega$. Under a superdiffusive scaling (and certain regularity conditions on the coefficients of $\mathcal{A}_\varepsilon^\omega$), they have obtained that the homogenized process is a pure-jump Lévy process (it is generated by a purely non-local operator). Related to this, the homogenization of a diffusion with jumps generated by

$$\mathcal{A}_\varepsilon f(x) := \frac{1}{\varepsilon^{\alpha_0}} \int_{\mathbb{R}^d} \left(f(\varepsilon y + x) - f(x) - \varepsilon \sum_{i=1}^d y_i \frac{\partial f(x)}{\partial x_i} 1_{B(0,1)}(y) \right) \frac{\beta(x/\varepsilon)}{|y|^{d+\alpha(x/\varepsilon)}} dy, \quad (1.6)$$

where $\alpha : \mathbb{R}^d \rightarrow (0, 2)$ and $\beta : \mathbb{R}^d \rightarrow (0, \infty)$ are periodic, continuously differentiable and $\{x \in \mathbb{R}^d : \alpha(x) = \alpha_0 := \inf_{x \in \mathbb{R}^d} \alpha(x)\}$ has positive Lebesgue measure, has been considered in [Fra06, Fra07] (by probabilistic methods) and, in the case of constant parameter $\alpha(x)$, in [Ari09] and [Sch10] (by an analytic method, two-scale convergence method, which strongly relies on the fact that the

parameter $\alpha(x)$ is constant, scaling property of the underlying jump kernel and regularity properties of the solution of the corresponding boundary value problem). Note that in this situation the jump kernel $\nu(x, dy) := \beta(x)|y|^{-d-\alpha(x)}dy$ does not satisfy anymore the relation in (1.2). Accordingly, the superdiffusive scaling $\varepsilon^{-\alpha_0}$ is used and the corresponding homogenized process cannot be anymore a Brownian motion (the jumps are too big) but it is a pure-jump Lévy process (stable Lévy-process) generated by

$$\mathcal{A}_0 f(x) := \int_{\mathbb{R}^d} \left(f(y+x) - f(x) - \sum_{i=1}^d y_i \frac{\partial f(x)}{\partial x_i} 1_{B(0,1)}(y) \right) \frac{\beta_0}{|y|^{d+\alpha_0}} dy$$

for some constant $\beta_0 > 0$. Finally, let us also remark that the homogenization of the operator in (1.6) (with constant parameter $\alpha(x)$) has been also discussed in: (i) the quasi-periodic and almost periodic setting (see [Ari01]) and (ii) the case when $\{\beta(x)\}_{x \in \mathbb{R}^d}$ forms a stationary ergodic family defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with ergodic group of measure preserving transformations $\{\tau_x\}_{x \in \mathbb{R}^d}$ acting on $(\Omega, \mathcal{F}, \mathbb{P})$ (see [Sch13]). Again, the techniques and methods used to resolve these problems strongly rely on the fact that the parameter $\alpha(x)$ is constant, scaling property of the underlying jump kernel and regularity properties of the solution of the corresponding boundary value problem.

The main techniques used in [BLP78] and [Bha85] (which strongly rely on the fact that the underlying process is a diffusion with smooth coefficients) are based on proving the convergence as $\varepsilon \searrow 0$ of finite-dimensional distributions of the diffusion $\{F_t^\varepsilon\}_{t \geq 0}$ and functional central limit theorems for stationary ergodic sequences constructed from this process. On the other hand, our approach is through the characteristics of semimartingales. More precisely, to determine the homogenized process, we reduce the problem to the convergence of semimartingale characteristics of the process $\{F_t^\varepsilon\}_{t \geq 0}$. Namely, by using the facts that $\{F_t^\varepsilon\}_{t \geq 0}$ is a semimartingale whose characteristics are given in terms of the coefficients of the corresponding generator \mathcal{A}_ε (see [Sch98b, Lemma 3.1 and Theorem 3.5]) and the regularity assumptions imposed upon the coefficients $c(x)$ and $\nu(x, dy)$, we show that the characteristics of $\{F_t^\varepsilon\}_{t \geq 0}$ as $\varepsilon \searrow 0$ converge (in probability) to the characteristics of a certain Brownian motion, which, according to [JS03, Theorem VIII.2.17], proves the desired result.

The remainder of the paper is organized as follows. In Section 2, we give some preliminaries on diffusions with jumps and in Section 3 we state the main result of the paper, Theorem 3.1. In Section 4, we provide some auxiliary results concerning periodic diffusions with jumps, in Section 5 we prove Theorem 3.1 and in Section 6 we discuss the non-zero drift case. Finally, in Section 7, we give an application of the presented homogenization results to the long-time behavior (transience, recurrence and ergodicity) of periodic diffusions with small jumps.

2 Preliminaries on Diffusions with Jumps

Let $(\Omega, \mathcal{F}, \{\mathbb{P}_x\}_{x \in \mathbb{R}^d}, \{\mathcal{F}_t\}_{t \geq 0}, \{\theta_t\}_{t \geq 0}, \{M_t\}_{t \geq 0})$, denoted by $\{M_t\}_{t \geq 0}$ in the sequel, be a Markov process with state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $d \geq 1$ and $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ -algebra on \mathbb{R}^d . Due to the Markov property, the associated family of linear operators $\{P_t\}_{t \geq 0}$ on $B_b(\mathbb{R}^d)$ (the space of bounded and Borel measurable functions), defined by

$$P_t f(x) := \mathbb{E}_x[f(M_t)], \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad f \in B_b(\mathbb{R}^d),$$

forms a *semigroup* on the Banach space $(B_b(\mathbb{R}^d), \|\cdot\|_\infty)$, that is, $P_s \circ P_t = P_{s+t}$ and $P_0 = I$ for all $s, t \geq 0$. Here, \mathbb{E}_x stands for the expectation with respect to $\mathbb{P}_x(d\omega)$, $x \in \mathbb{R}^d$, and $\|\cdot\|_\infty$

denotes the supremum norm on the space $B_b(\mathbb{R}^d)$. Moreover, the semigroup $\{P_t\}_{t \geq 0}$ is *contractive* ($\|P_t f\|_\infty \leq \|f\|_\infty$ for all $t \geq 0$ and all $f \in B_b(\mathbb{R}^d)$) and *positivity preserving* ($P_t f \geq 0$ for all $t \geq 0$ and all $f \in B_b(\mathbb{R}^d)$ satisfying $f \geq 0$). The *infinitesimal generator* $(\mathcal{A}^b, \mathcal{D}_{\mathcal{A}^b})$ of the semigroup $\{P_t\}_{t \geq 0}$ (or of the process $\{M_t\}_{t \geq 0}$) is a linear operator $\mathcal{A}^b : \mathcal{D}_{\mathcal{A}^b} \rightarrow B_b(\mathbb{R}^d)$ defined by

$$\mathcal{A}^b f := \lim_{t \rightarrow 0} \frac{P_t f - f}{t}, \quad f \in \mathcal{D}_{\mathcal{A}^b} := \left\{ f \in B_b(\mathbb{R}^d) : \lim_{t \rightarrow 0} \frac{P_t f - f}{t} \text{ exists in } \|\cdot\|_\infty \right\}.$$

We call $(\mathcal{A}^b, \mathcal{D}_{\mathcal{A}^b})$ the *B_b-generator* for short. A Markov process $\{M_t\}_{t \geq 0}$ is said to be a *Feller process* if its corresponding semigroup $\{P_t\}_{t \geq 0}$ forms a *Feller semigroup*. This means that

- (i) $\{P_t\}_{t \geq 0}$ enjoys the *Feller property*, that is, $P_t(C_\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$ for all $t \geq 0$;
- (ii) $\{P_t\}_{t \geq 0}$ is *strongly continuous*, that is, $\lim_{t \rightarrow 0} \|P_t f - f\|_\infty = 0$ for all $f \in C_\infty(\mathbb{R}^d)$.

Here, $C_\infty(\mathbb{R}^d)$ denotes the space of continuous functions vanishing at infinity. Note that every Feller semigroup $\{P_t\}_{t \geq 0}$ can be uniquely extended to $B_b(\mathbb{R}^d)$ (see [Sch98a, Section 3]). For notational simplicity, we denote this extension again by $\{P_t\}_{t \geq 0}$. Also, let us remark that every Feller process possesses the strong Markov property and has càdlàg sample paths (see [Jac05, Theorems 3.4.19 and 3.5.14]). Further, in the case of Feller processes, we call $(\mathcal{A}^\infty, \mathcal{D}_{\mathcal{A}^\infty}) := (\mathcal{A}^b, \mathcal{D}_{\mathcal{A}^b} \cap C_\infty(\mathbb{R}^d))$ the *Feller generator* for short. Observe that, in this case, $\mathcal{D}_{\mathcal{A}^\infty} \subseteq C_\infty(\mathbb{R}^d)$ and $\mathcal{A}^\infty(\mathcal{D}_{\mathcal{A}^\infty}) \subseteq C_\infty(\mathbb{R}^d)$. If the set of smooth functions with compact support $C_c^\infty(\mathbb{R}^d)$ is contained in $\mathcal{D}_{\mathcal{A}^\infty}$, that is, if the Feller generator $(\mathcal{A}^\infty, \mathcal{D}_{\mathcal{A}^\infty})$ of the Feller process $\{M_t\}_{t \geq 0}$ satisfies

$$(C1) \quad C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}_{\mathcal{A}^\infty},$$

then, according to [Cou66, Theorem 3.4], $\mathcal{A}^\infty|_{C_c^\infty(\mathbb{R}^d)}$ is a *pseudo-differential operator*, that is, it can be written in the form

$$\mathcal{A}^\infty|_{C_c^\infty(\mathbb{R}^d)} f(x) = - \int_{\mathbb{R}^d} q(x, \xi) e^{i\langle \xi, x \rangle} \hat{f}(\xi) d\xi, \quad (2.1)$$

where $\hat{f}(\xi) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle \xi, x \rangle} f(x) dx$ denotes the Fourier transform of the function $f(x)$. The function $q : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ is called the *symbol* of the pseudo-differential operator. It is measurable and locally bounded in (x, ξ) and continuous and negative definite as a function of ξ . Hence, by [Jac01, Theorem 3.7.7], the function $\xi \mapsto q(x, \xi)$ has for each $x \in \mathbb{R}^d$ the following Lévy-Khintchine representation

$$q(x, \xi) = a(x) - i\langle \xi, b(x) \rangle + \frac{1}{2} \langle \xi, c(x) \xi \rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, y \rangle} + i\langle \xi, y \rangle 1_{B(0,1)}(y) \right) \nu(x, dy), \quad (2.2)$$

where $a(x)$ is a non-negative Borel measurable function, $b(x)$ is an \mathbb{R}^d -valued Borel measurable function, $c(x) := (c_{ij}(x))_{1 \leq i, j \leq d}$ is a symmetric non-negative definite $d \times d$ matrix-valued Borel measurable function and $\nu(x, dy)$ is a Borel kernel on $\mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d)$, called the *Lévy measure*, satisfying

$$\nu(x, \{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(x, dy) < \infty, \quad x \in \mathbb{R}^d.$$

The quadruple $(a(x), b(x), c(x), \nu(x, dy))$ is called the *Lévy quadruple* of the pseudo-differential operator $\mathcal{A}^\infty|_{C_c^\infty(\mathbb{R}^d)}$ (or of the symbol $q(x, \xi)$). Let us remark that the local boundedness of $q(x, \xi)$ implies the local boundedness of the corresponding coefficients (see [Sch98b, Lemma 2.1 and Remark 2.2]). In the sequel, we assume the following condition on the symbol $q(x, \xi)$:

(C2) $q(x, 0) = a(x) = 0$ for all $x \in \mathbb{R}^d$.

This condition is closely related to the conservation property of $\{M_t\}_{t \geq 0}$. Namely, under the assumption that the coefficients of $q(x, \xi)$ are uniformly bounded (which is certainly the case in the periodic setting), (C2) implies that $\{M_t\}_{t \geq 0}$ is *conservative*, that is, $\mathbb{P}_x(M_t \in \mathbb{R}^d) = 1$ for all $t \geq 0$ and all $x \in \mathbb{R}^d$ (see [Sch98a, Theorem 5.2]). Further, note that by combining (2.1), (2.2) and (C2), $\mathcal{A}^\infty|_{C_c^\infty(\mathbb{R}^d)}$ has a representation as an integro-differential operator (1.1). Also note that in the case when the symbol $q(x, \xi)$ does not depend on the variable $x \in \mathbb{R}^d$, $\{M_t\}_{t \geq 0}$ becomes a *Lévy process*, that is, a stochastic process with stationary and independent increments and càdlàg sample paths. Moreover, unlike Feller processes, every Lévy process is uniquely and completely characterized through its corresponding symbol (see [Sat99, Theorems 7.10 and 8.1]). According to this, it is not hard to check that every Lévy process satisfies conditions (C1) and (C2) (see [Sat99, Theorem 31.5]). Thus, the class of processes we consider in this paper contains a subclass of Lévy processes. Throughout this paper, the symbol $\{F_t\}_{t \geq 0}$ denotes a Feller process satisfying conditions (C1) and (C2). Such a process is called a *diffusion process with jumps*. If $\nu(x, dy) = 0$ for all $x \in \mathbb{R}^d$, then $\{F_t\}_{t \geq 0}$ is just called a *diffusion process*. Note that this definition agrees with the standard definition of diffusions (see [RW00]). For more on diffusions with jumps we refer the readers to the monograph [BSW13].

3 Main Results

Before stating the main results of this paper, we introduce some notation we need. Let $\tau := (\tau_1, \dots, \tau_d) \in (0, \infty)^d$ be fixed and let $\tau\mathbb{Z}^d := \tau_1\mathbb{Z} \times \dots \times \tau_d\mathbb{Z}$. For $x \in \mathbb{R}^d$, define

$$x_\tau := \{y \in \mathbb{R}^d : x - y \in \tau\mathbb{Z}^d\} \quad \text{and} \quad \mathbb{R}^d / \tau\mathbb{Z}^d := \{x_\tau : x \in \mathbb{R}^d\}.$$

Clearly, $\mathbb{R}^d / \tau\mathbb{Z}^d$ is obtained by identifying the opposite faces of $[0, \tau] := [0, \tau_1] \times \dots \times [0, \tau_d]$. Next, let $\Pi_\tau : \mathbb{R}^d \rightarrow [0, \tau]$, $\Pi_\tau(x) := x_\tau$, be the covering map. A function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is called τ -periodic if

$$f \circ \Pi_\tau(x) = f(x), \quad x \in \mathbb{R}^d.$$

For notational convenience, from now on we will omit the subscript τ and simply write x instead of x_τ . Now, we are in position to state the main results of this paper, the proofs of which are given in Section 5.

Theorem 3.1. *Let $\{F_t\}_{t \geq 0}$ be a d -dimensional diffusion with jumps with semigroup $\{P_t\}_{t \geq 0}$, symbol $q(x, \xi)$ and Lévy triplet $(0, c(x), \nu(x, dy))$, which satisfy:*

(C3) $\{F_t\}_{t \geq 0}$ possesses a transition density function $p(t, x, y)$, that is,

$$P_t f(x) = \int_{\mathbb{R}^d} f(y) p(t, x, y) dy, \quad t > 0, \quad x \in \mathbb{R}^d, \quad f \in B_b(\mathbb{R}^d),$$

such that $(x, y) \mapsto p(t, x, y)$ is continuous and $p(t, x, y) > 0$ for all $t > 0$ and all $x, y \in \mathbb{R}^d$;

(C4) the function $x \mapsto q(x, \xi)$ (or, equivalently, the coefficients $c(x)$ and $\nu(x, dy)$) is τ -periodic for all $\xi \in \mathbb{R}^d$;

(C5) $\sup_{x \in [0, \tau]} \int_{\mathbb{R}^d} |y|^2 \nu(x, dy) < \infty$;

$$(\mathbf{C6}) \quad \lim_{\varepsilon \rightarrow 0} \sup_{x \in [0, \tau]} \left| \int_{B(0, \varepsilon^{-1}) \setminus B(0, 1)} y_i \nu(x, dy) \right| = 0 \quad \text{for all } i = 1, \dots, d.$$

Then, for any initial distribution of $\{F_t\}_{t \geq 0}$,

$$\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0} \xrightarrow[\varepsilon \searrow 0]{d} \{W_t\}_{t \geq 0}. \quad (3.1)$$

Here, \xrightarrow{d} denotes the convergence in the space of càdlàg functions endowed with the Skorohod topology, $\{W_t\}_{t \geq 0}$ is a d -dimensional zero-drift Brownian motion starting from the origin and determined by a covariance matrix of the form

$$\Sigma := \left(\int_{[0, \tau]} c_{ij}(z) \pi(dz) + \int_{[0, \tau]} \int_{\mathbb{R}^d} y_i y_j \nu(z, dy) \pi(dz) \right)_{1 \leq i, j \leq d} \quad (3.2)$$

and $\pi(dx)$ is an invariant probability measure associated to the projection of $\{F_t\}_{t \geq 0}$, with respect to $\Pi_\tau(x)$, on $[0, \tau]$ (see Section 4 for details).

Remark 3.2. In Theorem 3.1 we assume that the drift term vanishes, which is a crucial assumption. Namely, as we commented in the first section, in [BLP78] and [Bha85] the authors have considered the homogenization of a d -dimensional diffusion $\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0}$ generated by a second-order elliptic operator (1.3) with not necessarily vanishing drift term. More precisely, by using different approaches (which strongly rely on the fact that the underlying process is a diffusion with smooth coefficients), they have proved that: (i) the family of diffusions $\{\varepsilon F_{\varepsilon^{-2}t} - \varepsilon^{-1} \bar{b}t\}_{t \geq 0}$, $\varepsilon > 0$, is tight and (ii) the finite-dimensional distributions of $\{\varepsilon F_{\varepsilon^{-2}t} - \varepsilon^{-1} \bar{b}t\}_{t \geq 0}$ converge to those of $\{\bar{W}_t\}_{t \geq 0}$, as $\varepsilon \searrow 0$. Recall, $\bar{b} = (\int_{[0, \tau]} b_i(x) \pi(dx))_{1 \leq i \leq d}$ and $\{\bar{W}_t\}_{t \geq 0}$ is a zero-drift Brownian motion determined by the covariance matrix given in (1.4). Note that, due to Prokhorov's theorem, (i) and (ii) are equivalent to

$$\{\varepsilon F_{\varepsilon^{-2}t} - \varepsilon^{-1} \bar{b}t\}_{t \geq 0} \xrightarrow[\varepsilon \searrow 0]{d} \{\bar{W}_t\}_{t \geq 0}. \quad (3.3)$$

On the other hand, the approach through the characteristics of semimartingales gives only sufficient conditions for the convergence in the Skorohod topology and it is too weak to deal with the non-zero drift case. Namely, by using this approach, in Section 6 we prove that the relation in (3.3) (for a diffusion with jumps satisfying (C3)-(C6)) holds if, and only if, $b(x) = \bar{b}$ (dx -a.e.).

Remark 3.3. Note that (non-degenerate) diffusions always satisfy the assumptions in (C3) (see [RW00] and [She91, Theorem A]). Hence, Theorem 3.1 generalizes the results related to diffusions (with zero-drift term), presented in [BLP78] and [Bha85].

Remark 3.4. In (C3) we assume the existence, continuity (in space variables) and strict positivity of the transition density function $p(t, x, y)$ of $\{F_t\}_{t \geq 0}$. According to [San14a, Theorem 2.6], under the assumptions that $q(x, \xi)$ has bounded coefficients (which is certainly the case in the periodic setting) and $q(x, \xi) = \operatorname{Re} q(x, \xi)$ for all $x, \xi \in \mathbb{R}^d$ (or, equivalently, $b(x) = 0$ and $\nu(x, dy)$ is a symmetric measure for all $x \in \mathbb{R}^d$), the existence of $p(t, x, y)$ follows from

$$\int_{\mathbb{R}^d} \exp \left[-t \inf_{x \in \mathbb{R}^d} q(x, \xi) \right] d\xi < \infty, \quad t > 0. \quad (3.4)$$

According to [Fri64] and [Sat99, Theorems 7.10 and 8.1], in the Lévy process and diffusion case, in order to ensure the existence of a transition density function, the assumption that the symbol is

real is not essential. Further, note that (3.4) (together with the assumptions that the symbol has bounded coefficients and it is real) also implies the continuity of $(x, y) \mapsto p(t, x, y)$ for all $t > 0$. First, under these assumptions, [SW13, Theorems 1.1 and 1.2] entail that

$$\sup_{x \in \mathbb{R}^d} \left| \mathbb{E}^x \left[e^{i\xi(F_t - x)} \right] \right| \leq \exp \left[-\frac{t}{16} \inf_{x \in \mathbb{R}^d} q(x, 2\xi) \right], \quad t > 0, \xi \in \mathbb{R}^d,$$

and that the function $x \mapsto \mathbb{E}^x \left[e^{i\xi(F_t - x)} \right]$ is continuous for all $\xi \in \mathbb{R}^d$. Consequently, [Sat99, Proposition 2.5] states that $p(t, x, y)$ exists and it is given by

$$p(t, x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi(y-x)} \mathbb{E}^x \left[e^{i\xi(F_t - x)} \right] d\xi, \quad t > 0, x, y \in \mathbb{R}^d.$$

Finally, the continuity of $(x, y) \mapsto p(t, x, y)$, $t > 0$, follows directly by employing the dominated convergence theorem. Observe that (3.4) is trivially satisfied in the case when the diffusion coefficient is uniformly elliptic. On the other hand, the strict positivity of the transition density function $p(t, x, y)$ is a more complex problem. In the Lévy process and diffusion case this problem has been considered in [BH80], [BRŽ96], [Fri64], [Sha69] and [She91]. In the general case, when $\{F_t\}_{t \geq 0}$ is a diffusion with jumps with bounded coefficients and real symbol satisfying (3.4), the best we were able to prove is given in [PS15, Proposition 6.1] and it reads as follows: $\{F_t\}_{t \geq 0}$ possesses a transition density function $p(t, x, y)$ such that for any $t_0 > 0$ and any $n \geq 1$ there exists $\varepsilon(t_0) > 0$ such that $p(t, x, y + x) > 0$ for all $t \in [nt_0, n(t_0 + 1)]$, all $x \in \mathbb{R}^d$ and all $y \in B(0, n\varepsilon(t_0))$.

As a direct consequence of Theorem 3.1 and [BSW13, Theorem 7.1] we get the following result.

Corollary 3.5. *Let $\{F_t\}_{t \geq 0}$ be a d -dimensional diffusion with jumps with semigroup $\{P_t\}_{t \geq 0}$ and symbol $q(x, \xi)$, satisfying conditions (C3)-(C6). Then, for $\varepsilon > 0$, $\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0}$ is a diffusion with jumps determined by symbol of the form $q^\varepsilon(x, \xi) := \varepsilon^{-2}q(x/\varepsilon, \varepsilon\xi)$. Further, let $\{P_t^\varepsilon\}_{t \geq 0}$ and $(\mathcal{A}_\varepsilon^\infty, \mathcal{D}_{\mathcal{A}_\varepsilon^\infty})$, $\varepsilon > 0$, and $\{P_t^0\}_{t \geq 0}$ and $(\mathcal{A}_0^\infty, \mathcal{D}_{\mathcal{A}_0^\infty})$ be the corresponding Feller semigroups and generators of $\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0}$, $\varepsilon > 0$, and $\{W_t\}_{t \geq 0}$, respectively. Then,*

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \leq t_0} \|P_t^\varepsilon f - P_t^0 f\|_\infty = 0, \quad t_0 \geq 0, f \in C_\infty(\mathbb{R}^d),$$

and for each $f \in C_c^\infty(\mathbb{R}^d)$ there exist functions $f_\varepsilon \in \mathcal{D}_{\mathcal{A}_\varepsilon^\infty}$, $\varepsilon > 0$, such that

$$\lim_{\varepsilon \rightarrow 0} (\|f_\varepsilon - f\|_\infty + \|\mathcal{A}_\varepsilon^\infty f_\varepsilon - \mathcal{A}_0^\infty f\|_\infty) = 0.$$

Definition 3.6. *Let $O \subseteq \mathbb{R}^d$, $d \geq 2$, be a connected open set. The set O satisfies the Poincaré cone condition at $x \in \partial O$ if there exists a cone C based at x with opening angle $\varphi > 0$, and $r > 0$ such that $C \cap B(x, r) \subseteq O^c$. Here, ∂O denotes the boundary of the set O .*

Theorem 3.7. *Let $\{F_t\}_{t \geq 0}$ be a d -dimensional diffusion with jumps satisfying conditions (C3)-(C6) and let $\{W_t\}_{t \geq 0}$ be a zero-drift Brownian motion determined by the covariance matrix Σ (given in (3.2)). Further, let $O \subseteq \mathbb{R}^d$, $d \geq 2$, be an open set such that ∂O has Lebesgue measure zero and O and $\text{Int } O^c$ (interior of the set O^c) are connected and satisfy the Poincaré cone condition on ∂O . If $d = 1$, just assume that $O \subseteq \mathbb{R}$ is open. For $\varepsilon > 0$, define $T^\varepsilon := \inf\{t > 0 : \varepsilon F_{\varepsilon^{-2}t} \notin O\}$ and $T^0 := \inf\{t > 0 : W_t \notin O\}$. Then, for any initial distribution of $\{F_t\}_{t \geq 0}$,*

$$\varepsilon F_{\varepsilon^{-2}T^\varepsilon} \xrightarrow[\varepsilon \searrow 0]{d} W_{T^0} \tag{3.5}$$

and

$$\{\varepsilon F_{\varepsilon^{-2}(t \wedge T^\varepsilon)}\}_{t \geq 0} \xrightarrow[\varepsilon \searrow 0]{d} \{W_{t \wedge T^0}\}_{t \geq 0}. \tag{3.6}$$

As a consequence of Theorem 3.7 and [Fri75, Theorem 6.5.1] we get the following well-known result.

Corollary 3.8. *Let $\{F_t\}_{t \geq 0}$ be a d -dimensional diffusion with vanishing drift coefficient and uniformly elliptic diffusion coefficient $c(x)$, satisfying conditions (C3) and (C4). Further, let $\{W_t\}_{t \geq 0}$ be a zero-drift Brownian motion determined by the covariance matrix Σ (given in (3.2)), let $O \subseteq \mathbb{R}^d$ be an open bounded set with C^2 boundary satisfying the assumptions from Theorem 3.7 and let $a : O \cup \partial O \rightarrow (-\infty, 0]$ be an arbitrary Hölder continuous function. For $\varepsilon > 0$, let $(\mathcal{A}_\varepsilon^\infty, \mathcal{D}_{\mathcal{A}_\varepsilon^\infty})$ and $(\mathcal{A}_0^\infty, \mathcal{D}_{\mathcal{A}_0^\infty})$ be the Feller generators of $\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0}$ and $\{W_t\}_{t \geq 0}$, respectively, that is,*

$$\mathcal{A}_\varepsilon^\infty f(x) = \frac{1}{2} \sum_{i,j=1}^d c_{ij}(x/\varepsilon) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \quad \text{and} \quad \mathcal{A}_0^\infty f(x) = \frac{1}{2} \sum_{i,j=1}^d \int_{[0,\tau]} c_{ij}(z) \pi(dz) \frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

Then, for any $\varepsilon \geq 0$, any Hölder continuous function $f : O \cup \partial O \rightarrow \mathbb{R}$ and any continuous function $g : \partial O \rightarrow \mathbb{R}$, there exists a unique solution $u_\varepsilon(x)$ to the following Dirichlet problem

$$\begin{aligned} \mathcal{A}_\varepsilon^\infty u_\varepsilon(x) + a(x)u_\varepsilon(x) &= f(x), & x \in O \\ u_\varepsilon(x) &= g(x), & x \in \partial O. \end{aligned}$$

Moreover, $u_\varepsilon(x)$ converges (pointwise) to $u(x)$ as $\varepsilon \searrow 0$.

4 Auxiliary Results

We start this section with the following observation. Let $\{F_t\}_{t \geq 0}$ be a d -dimensional diffusion with jumps with transition density function $p(t, x, y)$ and Lévy triplet $(b(x), c(x), \nu(x, dy))$. Then, the coefficients $b(x)$, $c(x)$ and $\nu(x, dy)$ are τ -periodic if, and only if, the function $x \mapsto p(t, x, x + y)$ is τ -periodic for all $t > 0$ and all $y \in \mathbb{R}^d$. The sufficiency follows directly from [Jac01, the proof of Theorem 4.5.21]. To prove the necessity, first recall that there exists a suitable enlargement of the stochastic basis $(\Omega, \mathcal{F}, \{\mathbb{P}_x\}_{x \in \mathbb{R}^d}, \{\mathcal{F}_t\}_{t \geq 0}, \{\theta_t\}_{t \geq 0})$, say $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathbb{P}}_x\}_{x \in \mathbb{R}^d}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \{\tilde{\theta}_t\}_{t \geq 0})$, on which $\{F_t\}_{t \geq 0}$ is the solution to the following stochastic differential equation

$$\begin{aligned} F_t &= x + \int_0^t b(F_{s-}) ds + \int_0^t c(F_{s-}) d\tilde{W}_s \\ &\quad + \int_0^t \int_{\mathbb{R} \setminus \{0\}} k(F_{s-}, z) 1_{\{u: |k(F_{s-}, u)| \leq 1\}}(z) \left(\tilde{\mu}(\cdot, ds, dz) - ds \tilde{N}(dz) \right) \\ &\quad + \int_0^t \int_{\mathbb{R} \setminus \{0\}} k(F_{s-}, z) 1_{\{u: |k(F_{s-}, u)| > 1\}}(z) \tilde{\mu}(\cdot, ds, dz), \end{aligned} \tag{4.1}$$

where $\{\tilde{W}_t\}_{t \geq 0}$ is a d -dimensional Brownian motion, $\tilde{\mu}(\omega, ds, dz)$ is a Poisson random measure with compensator (dual predictable projection) $ds \tilde{N}(dz)$ and $k : \mathbb{R}^d \times \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}^d$ is a Borel measurable function satisfying

$$\tilde{\mu}(\omega, ds, k(F_{s-}(\omega), \cdot) \in dy) = \sum_{s: \Delta F_s(\omega) \neq 0} \delta_{(s, \Delta F_s(\omega))}(ds, dy)$$

and

$$ds \tilde{N}(k(F_{s-}(\omega), \cdot) \in dy) = ds \nu(F_{s-}(\omega), dy)$$

(see [Sch98b, Theorem 3.5] and [CJ81, Theorem 3.33]). Further, $\{F_t\}_{t \geq 0}$ has the same transition function on the starting and enlarged stochastic basis. Thus, because of the τ -periodicity of $b(x)$, $c(x)$, and $\nu(x, dy)$, directly from (4.1) we read that $\mathbb{P}_{x+\tau}(F_t \in dy) = \mathbb{P}_x(F_t + \tau \in dy)$ for all $t \geq 0$ and all $x \in \mathbb{R}^d$, which proves the assertion.

Now, according to the above observation, we easily deduce that if $\{F_t\}_{t \geq 0}$ is a d -dimensional diffusion with jumps with semigroup $\{P_t\}_{t \geq 0}$, transition density function $p(t, x, y)$ and τ -periodic Lévy triplet $(b(x), c(x), \nu(x, dy))$, then $\{P_t\}_{t \geq 0}$ preserves the class of all bounded Borel measurable τ -periodic functions, that is, the function $x \mapsto P_t f(x)$ is τ -periodic for all $t \geq 0$ and all τ -periodic $f \in B_b(\mathbb{R}^d)$. Consequently, [Kol11, Proposition 3.8.3] entails that $F_t^\tau := \Pi_\tau(F_t)$, $t \geq 0$, is a Markov process on $([0, \tau], \mathcal{B}([0, \tau]))$ with positivity preserving contraction semigroup $\{P_t^\tau\}_{t \geq 0}$ (on the space $(B_b([0, \tau]), \|\cdot\|_\infty)$) given by

$$P_t^\tau f(x) := \mathbb{E}_x^\tau[f(F_t^\tau)] = \int_{[0, \tau]} f(y) \mathbb{P}_x^\tau(F_t^\tau \in dy), \quad t \geq 0, \quad x \in [0, \tau], \quad f \in B_b([0, \tau]),$$

where

$$\mathbb{P}_x^\tau(F_t^\tau \in dy) := \sum_{k \in \tau\mathbb{Z}^d} \mathbb{P}_{z_x}(F_t - k \in dz_y), \quad t > 0, \quad x, y \in [0, \tau], \quad (4.2)$$

and z_x and z_y are arbitrary points in $\Pi_\tau^{-1}(\{x\})$ and $\Pi_\tau^{-1}(\{y\})$, respectively. From (4.2) we automatically conclude that $\{F_t^\tau\}_{t \geq 0}$ has a transition density function $p^\tau(t, x, y)$ which is given by

$$p^\tau(t, x, y) = \sum_{k \in \tau\mathbb{Z}^d} p(t, x, y + k), \quad t > 0, \quad x, y \in [0, \tau].$$

In particular, if, in addition to the τ -periodicity of the coefficients, $\{F_t\}_{t \geq 0}$ satisfies **(C3)**, then

$$\inf_{x, y \in [0, \tau]} p^\tau(t, x, y) > 0, \quad t > 0, \quad (4.3)$$

which suggests that it is reasonable to expect that $\{F_t^\tau\}_{t \geq 0}$ is ergodic.

Recall, a probability measure $\pi(dx)$ on a measurable space (S, \mathcal{S}) is *invariant* for a Markov process $(\Omega, \mathcal{F}, \{\mathbb{P}_x\}_{x \in S}, \{\mathcal{F}_t\}_{t \geq 0}, \{\theta_t\}_{t \geq 0}, \{M_t\}_{t \geq 0})$, denoted by $\{M_t\}_{t \geq 0}$ in the sequel, if

$$\int_S \mathbb{P}_x(M_t \in B) \pi(dx) = \pi(B), \quad t \geq 0, \quad B \in \mathcal{S}.$$

A set $B \in \mathcal{F}$ is said to be *shift-invariant* if $\theta_t^{-1}B = B$ for all $t \geq 0$. The *shift-invariant* σ -algebra \mathcal{I} is a collection of all such shift-invariant sets. The process $\{M_t\}_{t \geq 0}$ is said to be *ergodic* if it possesses an invariant probability measure $\pi(dx)$ and if \mathcal{I} is trivial with respect to $\mathbb{P}_\pi(d\omega)$, that is, $\mathbb{P}_\pi(B) = 0$ or 1 for every $B \in \mathcal{I}$. Here, for a probability measure $\mu(dx)$ on \mathcal{S} , $\mathbb{P}_\mu(d\omega)$ is defined as $\mathbb{P}_\mu(d\omega) := \int_S \mathbb{P}_x(d\omega) \mu(dx)$. The process $\{M_t\}_{t \geq 0}$ is said to be *strongly ergodic* if it possesses an invariant probability measure $\pi(dx)$ and if

$$\lim_{t \rightarrow \infty} \|\mathbb{P}_x(M_t \in \cdot) - \pi(\cdot)\|_{TV} = 0, \quad x \in S, \quad (4.4)$$

where $\|\cdot\|_{TV}$ denotes the total variation norm on the space of signed measures on \mathcal{S} . Clearly, an invariant probability measure of a strongly ergodic Markov processes is unique. In general, the strong ergodicity implies ergodicity (see [Bha82, Proposition 2.5]). On the other hand, ergodicity does not necessarily imply strong ergodicity (for example, see [PS15]). Sufficient condition for the strong ergodicity of Markov processes is given through the well-known Doeblin's condition. More

precisely, a Markov process $\{M_t\}_{t \geq 0}$ satisfies *Doebelin's condition* if there exist a probability measure $\mu(dx)$ on \mathcal{S} , $t_0 > 0$ and $\varepsilon \in (0, 1)$, such that $\mathbb{P}_x(M_{t_0} \in B) \leq 1 - \varepsilon$ whenever $\mu(B) \leq \varepsilon$, for all $x \in \mathcal{S}$. Moreover, in this case, the convergence in (4.4) is uniformly exponentially fast (see [Doo53, Theorem VI.2.1]). Now, by fixing $t_0 > 0$ and

$$\varepsilon \in \left(0, 1 \wedge \frac{|\tau| \inf_{x, y \in [0, \tau]} p^\tau(t_0, x, y)}{1 + \inf_{x, y \in [0, \tau]} p^\tau(t_0, x, y)}\right)$$

and taking $\mu(dx) := dx$, where $|\tau| := \tau_1 \cdots \tau_d$, (4.3) immediately entails that $\{F_t^\tau\}_{t \geq 0}$ satisfies Doeblin's condition. Hence, $\{F_t^\tau\}_{t \geq 0}$ possesses a unique invariant probability measure $\pi(dx)$ satisfying

$$\sup \{|P_t^\tau 1_B(x) - \pi(B)| : x \in [0, \tau], B \in \mathcal{B}([0, \tau])\} \leq C e^{-ct} \quad (4.5)$$

for all $t \geq 0$ and some universal constants $c > 0$ and $C > 0$. Let us remark that the exponential decay in (4.5) will be crucial in the proof of Theorem 3.1.

5 Proofs of the Main Results

Before the proof of Theorem 3.1, let us recall the notion of characteristics of a semimartingale (see [JS03]). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{S_t\}_{t \geq 0})$, denoted by $\{S_t\}_{t \geq 0}$ in the sequel, be a d -dimensional semimartingale and let $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a truncation function (that is, a continuous and bounded function which satisfies $h(x) = x$ in a neighborhood of the origin). Define,

$$\check{S}(h)_t := \sum_{s \leq t} (\Delta S_s - h(\Delta S_s)), \quad t \geq 0, \quad \text{and} \quad S(h)_t := S_t - \check{S}(h)_t, \quad t \geq 0,$$

where the process $\{\Delta S_t\}_{t \geq 0}$ is defined by $\Delta S_t := S_t - S_{t-}$ and $\Delta S_0 := S_0$. The process $\{S(h)_t\}_{t \geq 0}$ is a *special semimartingale*, that is, it admits a unique decomposition

$$S(h)_t = S_0 + M(h)_t + B(h)_t, \quad (5.1)$$

where $\{M(h)_t\}_{t \geq 0}$ is a local martingale and $\{B(h)_t\}_{t \geq 0}$ is a predictable process of bounded variation.

Definition 5.1. Let $\{S_t\}_{t \geq 0}$ be a semimartingale and let $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a truncation function. Furthermore, let $\{B(h)_t\}_{t \geq 0}$ be the predictable process of bounded variation appearing in (5.1), let $N(\omega, ds, dy)$ be the compensator of the jump measure

$$\mu(\omega, ds, dy) := \sum_{s: \Delta S_s(\omega) \neq 0} \delta_{(s, \Delta S_s(\omega))}(ds, dy)$$

of the process $\{S_t\}_{t \geq 0}$ and let $\{C_t\}_{t \geq 0} = \{(C_t^{ij})_{1 \leq i, j \leq d}\}_{t \geq 0}$ be the quadratic co-variation process for $\{S_t^c\}_{t \geq 0}$ (continuous martingale part of $\{S_t\}_{t \geq 0}$), that is, $C_t^{ij} = \langle S_t^{i,c}, S_t^{j,c} \rangle$. Then (B, C, N) is called the characteristics of the semimartingale $\{S_t\}_{t \geq 0}$ (relative to $h(x)$). In addition, by defining $\tilde{C}(h)_t^{ij} := \langle M(h)_t^i, M(h)_t^j \rangle$, $i, j = 1, \dots, d$, where $\{M(h)_t\}_{t \geq 0}$ is the local martingale appearing in (5.1), (B, \tilde{C}, N) is called the modified characteristics of the semimartingale $\{S_t\}_{t \geq 0}$ (relative to $h(x)$).

Now, we prove Theorem 3.1. We follow the idea from [Fra06, Theorem 1] (see also [San14b, Theorem 1.2]).

Proof of Theorem 3.1. First, recall that, for any initial distribution $\mu(dx)$ of $\{F_t\}_{t \geq 0}$, $\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0}$, $\varepsilon > 0$, are \mathbb{P}_μ - semimartingales (with respect to the natural filtration generated by $\{F_t\}_{t \geq 0}$) whose (modified) characteristics (relative to $h(x)$) are given by

$$\begin{aligned} B(h)_t^{\varepsilon,i} &= \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{R}^d} (h_i(\varepsilon y) - \varepsilon y_i 1_{B(0,1)}(y)) \nu(F_{\varepsilon^{-2}s}, dy) ds, \\ C_t^{\varepsilon,ij} &= \int_0^t c_{ij}(F_{\varepsilon^{-2}s}) ds, \\ N^\varepsilon(ds, B) &= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} 1_B(\varepsilon y) \nu(F_{\varepsilon^{-2}s}, dy) ds, \\ \tilde{C}(h)_t^{\varepsilon,ij} &= \int_0^t c_{ij}(F_{\varepsilon^{-2}s}) ds + \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{R}^d} h_i(\varepsilon y) h_j(\varepsilon y) \nu(F_{\varepsilon^{-2}s}, dy) ds, \end{aligned}$$

$t \geq 0$, $B \in \mathcal{B}(\mathbb{R}^d)$, $i, j = 1, \dots, d$, (see [Sch98b, Lemma 3.2 and Theorem 3.5] and [JS03, Proposition II.2.17]). Now, according to [JS03, Theorem VIII.2.17], in order to prove the desired convergence it suffices to prove that

$$B(h)_t^{\varepsilon,i} \xrightarrow[\varepsilon \searrow 0]{L^2(\mathbb{P}_\mu, \Omega)} 0 \quad (5.2)$$

for all $t \geq 0$ and all $i = 1, \dots, d$,

$$\int_0^t \int_{\mathbb{R}^d} g(y) N^\varepsilon(ds, dy) \xrightarrow[\varepsilon \searrow 0]{L^2(\mathbb{P}_\mu, \Omega)} 0 \quad (5.3)$$

for all $t \geq 0$ and all $g \in C_b(\mathbb{R}^d)$ vanishing in a neighborhood of the origin, and

$$\tilde{C}(h)_t^{\varepsilon,ij} \xrightarrow[\varepsilon \searrow 0]{L^2(\mathbb{P}_\mu, \Omega)} t \Sigma^{ij} \quad (5.4)$$

for all $t \geq 0$ and all $i, j = 1, \dots, d$, where Σ is given in (3.2).

First, we prove the relation in (5.2). For all $\varepsilon > 0$ small enough, we have

$$B(h)_t^{\varepsilon,i} = \frac{1}{\varepsilon} \int_0^t \int_{B(0, \varepsilon^{-1}\delta) \setminus B(0,1)} y_i \nu(F_{\varepsilon^{-2}s}, dy) ds + \frac{1}{\varepsilon^2} \int_0^t \int_{B^c(0, \varepsilon^{-1}\delta)} h_i(\varepsilon y) \nu(F_{\varepsilon^{-2}s}, dy) ds,$$

where $\delta > 0$ is such that $h(y) = y$ on $B(0, \delta)$. Now, define

$$\begin{aligned} U_1^{\varepsilon,i}(x) &:= \frac{1}{\varepsilon} \int_{B(0, \varepsilon^{-1}\delta) \setminus B(0,1)} y_i \left(\nu(x, dy) - \int_{[0, \tau]} \nu(z, dy) \pi(dz) \right), \\ U_2^{\varepsilon,i}(x) &:= \frac{1}{\varepsilon^2} \int_{B^c(0, \varepsilon^{-1}\delta)} h_i(\varepsilon y) \left(\nu(x, dy) - \int_{[0, \tau]} \nu(z, dy) \pi(dz) \right). \end{aligned}$$

Clearly, for all $\varepsilon > 0$ small enough, $U_k^{\varepsilon,i}(x)$, $k = 1, 2$, are bounded, τ -periodic and satisfy $U_k^{\varepsilon,i}(F_t) = U_k^{\varepsilon,i}(F_t^\tau)$, $t \geq 0$, and

$$\int_{[0, \tau]} U_k^{\varepsilon,i}(x) \pi(dx) = 0.$$

Thus, by the Markov property, (4.2) and (4.5),

$$\begin{aligned}
\mathbb{E}_\mu \left[\left(\int_0^t U_1^{\varepsilon,i}(F_{\varepsilon^{-2}s}) ds \right)^2 \right] &= \mathbb{E}_\mu^\tau \left[\left(\int_0^t U_1^{\varepsilon,i}(F_{\varepsilon^{-2}s}^\tau) ds \right)^2 \right] \\
&= 2 \int_0^t \int_0^s \mathbb{E}_\mu^\tau \left[U_1^{\varepsilon,i}(F_{\varepsilon^{-2}s}^\tau) U_1^{\varepsilon,i}(F_{\varepsilon^{-2}u}^\tau) \right] dud s \\
&= 2 \int_0^t \int_0^s \mathbb{E}_\mu^\tau \left[P_{\varepsilon^{-2}(s-u)}^\tau U_1^{\varepsilon,i}(F_{\varepsilon^{-2}u}^\tau) U_1^{\varepsilon,i}(F_{\varepsilon^{-2}u}^\tau) \right] \\
&\leq 2C \|U_1^{\varepsilon,i}\|_\infty^2 \int_0^t \int_0^s e^{-c\varepsilon^{-2}(s-u)} dud s \\
&\leq \frac{4C\varepsilon^2 t}{c} \|U_1^{\varepsilon,i}\|_\infty^2 \\
&= \frac{4Ct}{c} \sup_{x \in [0,\tau]} \left(\int_{B(0,\varepsilon^{-1}\delta) \setminus B(0,1)} y_i \left(\nu(x, dy) - \int_{[0,\tau]} \nu(z, dy) \pi(dz) \right) \right)^2 \\
&\leq \frac{16Ct}{c} \sup_{x \in [0,\tau]} \left| \int_{B(0,\varepsilon^{-1}\delta) \setminus B(0,1)} y_i \nu(x, dy) \right|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}_\mu \left[\left(\int_0^t U_2^{\varepsilon,i}(F_{\varepsilon^{-2}s}) ds \right)^2 \right] &= 2 \int_0^t \int_0^s \mathbb{E}_\mu^\tau \left[P_{\varepsilon^{-2}(s-u)}^\tau U_2^{\varepsilon,i}(F_{\varepsilon^{-2}u}^\tau) U_2^{\varepsilon,i}(F_{\varepsilon^{-2}u}^\tau) \right] \\
&\leq 2C \|U_2^{\varepsilon,i}\|_\infty^2 \int_0^t \int_0^s e^{-c\varepsilon^{-2}(s-u)} dud s \\
&\leq \frac{4C \|h\|_\infty^2 t}{c\varepsilon^2} \sup_{x \in [0,\tau]} \left(\nu(x, B^c(0, \varepsilon^{-1}\delta)) - \int_{[0,\tau]} \nu(z, B^c(0, \varepsilon^{-1}\delta)) \pi(dz) \right)^2 \\
&\leq \frac{16C \|h\|_\infty^2 \varepsilon^2 t}{c\delta^4} \sup_{x \in [0,\tau]} \left(\int_{B^c(0, \varepsilon^{-1}\delta)} |y|^2 \nu(x, dy) \right)^2,
\end{aligned}$$

where in the final step we used **(C5)**. Now, for all $\varepsilon > 0$ small enough,

$$\begin{aligned}
&\left(\mathbb{E}_\mu \left[\left(B(h)_t^{\varepsilon,i} \right)^2 \right] \right)^{\frac{1}{2}} \\
&\leq \left(\mathbb{E}_\mu \left[\left(\int_0^t U_1^{\varepsilon,i}(F_{\varepsilon^{-2}s}) ds \right)^2 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E}_\mu \left[\left(\frac{t}{\varepsilon} \int_{[0,\tau]} \int_{B(0,\varepsilon^{-1}\delta) \setminus B(0,1)} y_i \nu(z, dy) \pi(dz) \right)^2 \right] \right)^{\frac{1}{2}} \\
&\quad + \left(\mathbb{E}_\mu \left[\left(\int_0^t U_2^{\varepsilon,i}(F_{\varepsilon^{-2}s}) ds \right)^2 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E}_\mu \left[\left(\frac{t}{\varepsilon^2} \int_{[0,\tau]} \int_{B^c(0, \varepsilon^{-1}\delta)} h_i(\varepsilon y) \nu(z, dy) \pi(dz) \right)^2 \right] \right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{4C^{1/2}t^{1/2}}{c^{1/2}} \sup_{x \in [0, \tau]} \left| \int_{B(0, \varepsilon^{-1}\delta) \setminus B(0, 1)} y_i \nu(x, dy) \right| + \frac{t}{\varepsilon} \int_{[0, \tau]} \int_{B^c(0, \varepsilon^{-1}\delta)} y_i \nu(z, dy) \pi(dz) \\
&\quad + \frac{4C^{1/2} \|h\|_{\infty} \varepsilon t^{1/2}}{c^{1/2} \delta^2} \sup_{x \in [0, \tau]} \int_{B^c(0, \varepsilon^{-1}\delta)} |y|^2 \nu(x, dy) + \frac{\|h\|_{\infty} t}{\delta^2} \sup_{x \in [0, \tau]} \int_{B^c(0, \varepsilon^{-1}\delta)} |y|^2 \nu(x, y) \\
&\leq \frac{4C^{1/2}t^{1/2}}{c^{1/2}} \sup_{x \in [0, \tau]} \left| \int_{B(0, \varepsilon^{-1}\delta) \setminus B(0, 1)} y_i \nu(x, dy) \right| + \frac{t}{\delta} \sup_{x \in [0, \tau]} \int_{B^c(0, \varepsilon^{-1}\delta)} |y|^2 \nu(x, dy) \\
&\quad + \frac{4C^{1/2} \|h\|_{\infty} \varepsilon t^{1/2}}{c^{1/2} \delta^2} \sup_{x \in [0, \tau]} \int_{B^c(0, \varepsilon^{-1}\delta)} |y|^2 \nu(x, dy) + \frac{\|h\|_{\infty} t}{\delta^2} \sup_{x \in \mathbb{R}^d} \int_{B^c(0, \varepsilon^{-1}\delta)} |y|^2 \nu(x, y),
\end{aligned}$$

where in the second step, due to **(C6)**, we used the fact that

$$\int_{B^c(0, 1)} y_i \nu(x, dy) = 0, \quad x \in [0, \tau].$$

Thus, (5.2) follows by employing **(C5)** and **(C6)**.

Next, let us prove the relation in (5.3). Fix $g \in C_b(\mathbb{R}^d)$ which vanishes on $B(0, \delta)$, for some $\delta > 0$. Further, define

$$V^{\varepsilon}(x) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} g(\varepsilon y) \left(\nu(x, dy) - \int_{[0, \tau]} \nu(z, dy) \pi(dz) \right).$$

Clearly, for any $\varepsilon > 0$, $V^{\varepsilon}(x)$ has the same properties as $U_1^{\varepsilon, i}(x)$ and $U_2^{\varepsilon, i}(x)$. Thus, similarly as above, we get

$$\begin{aligned}
\mathbb{E}_{\mu} \left[\left(\int_0^t V^{\varepsilon}(F_{\varepsilon^{-2}s}) ds \right)^2 \right] &\leq 2C \|V^{\varepsilon, i}\|_{\infty}^2 \int_0^t \int_0^s e^{-c\varepsilon^{-2}(s-u)} du ds \\
&\leq \frac{4C \|g\|_{\infty}^2 t}{c\varepsilon^2} \sup_{x \in [0, \tau]} \left(\nu(x, B^c(0, \varepsilon^{-1}\delta)) - \int_{[0, \tau]} \nu(z, B^c(0, \varepsilon^{-1}\delta)) \pi(dz) \right)^2 \\
&\leq \frac{16C \|g\|_{\infty}^2 \varepsilon^2 t}{c\delta^4} \sup_{x \in [0, \tau]} \left(\int_{B^c(0, \varepsilon^{-1}\delta)} |y|^2 \nu(x, dy) \right)^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\left(\mathbb{E}_{\mu} \left[\left(\int_0^t \int_{\mathbb{R}^d} g(y) N^{\varepsilon}(ds, dy) \right)^2 \right] \right)^{\frac{1}{2}} \\
&\leq \left(\mathbb{E}_{\mu} \left[\left(\int_0^t V^{\varepsilon}(\bar{F}_{\varepsilon^{-2}s}) ds \right)^2 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E}_{\mu} \left[\left(\frac{t}{\varepsilon^2} \int_{[0, \tau]} \int_{\mathbb{R}^d} g(\varepsilon y) \nu(z, dy) \pi(dz) \right)^2 \right] \right)^{\frac{1}{2}} \\
&\leq \frac{4C^{1/2} \|g\|_{\infty} \varepsilon t^{1/2}}{c^{1/2} \delta^2} \sup_{x \in [0, \tau]} \int_{B^c(0, \varepsilon^{-1}\delta)} |y|^2 \nu(x, dy) + \frac{\|g\|_{\infty} t}{\delta^2} \int_{B^c(0, \varepsilon^{-1}\delta)} |y|^2 \nu(x, dy),
\end{aligned}$$

which together with **(C5)** proves (5.3).

Finally, let us prove the relation in (5.4). We have

$$\begin{aligned}\tilde{C}(h)_t^{\varepsilon,ij} &= \int_0^t c_{ij}(F_{\varepsilon^{-2}s})ds + \int_0^t \int_{B(0,\varepsilon^{-1}\delta)} y_i y_j \nu(F_{\varepsilon^2 s}, dy) ds \\ &\quad + \frac{1}{\varepsilon^2} \int_0^t \int_{B^c(0,\varepsilon^{-1}\delta)} h_i(\varepsilon y) h_j(\varepsilon y) \nu(F_{\varepsilon^2 s}, dy) ds,\end{aligned}$$

where $\delta > 0$ is such that $h(y) = y$ on $B(0, \delta)$. Now, define

$$\begin{aligned}W_1^{\varepsilon,ij}(x) &:= c_{ij}(x) - \int_{[0,\tau]} c_{ij}^\tau(z) \pi(dz), \\ W_2^{\varepsilon,ij}(x) &:= \int_{B(0,\varepsilon^{-1}\delta)} y_i y_j \left(\nu(x, dy) - \int_{[0,\tau]} \nu(z, dy) \pi(dz) \right), \\ W_3^{\varepsilon,ij}(x) &:= \frac{1}{\varepsilon^2} \int_{B^c(0,\varepsilon^{-1}\delta)} h^i(\varepsilon y) h^j(\varepsilon y) \left(\nu(x, dy) - \int_{[0,\tau]} \nu(z, dy) \pi(dz) \right).\end{aligned}$$

Again, for any $\varepsilon > 0$, $W_k^{\varepsilon,ij}(x)$, $k = 1, 2, 3$, are bounded, τ -periodic and satisfy $W_k^{\varepsilon,ij}(F_t) = W_k^{\varepsilon,ij}(F_t^\tau)$, $t \geq 0$, and

$$\int_{[0,\tau]} W_k^{\varepsilon,ij}(x) \pi(dx) = 0.$$

Therefore, by using the same arguments as above, from (4.2) and (4.5) we get

$$\left(\mathbb{E}_\mu \left[\left(\int_0^t W_1^{\varepsilon,ij}(F_{\varepsilon^{-2}s}) ds \right)^2 \right] \right)^{\frac{1}{2}} \leq \frac{4C^{1/2} \varepsilon t^{1/2}}{c^{1/2}} \|c_{ij}\|_\infty, \quad (5.5)$$

$$\begin{aligned}&\left(\mathbb{E}_\mu \left[\left(\int_0^t \int_{B(0,\varepsilon^{-1}\delta)} y_i y_j \nu(F_{\varepsilon^{-2}s}, dy) ds - t \int_{[0,\tau]} \int_{\mathbb{R}^d} y_i y_j \nu(z, dy) \pi(dz) \right)^2 \right] \right)^{\frac{1}{2}} \\ &\leq \left(\mathbb{E}_\mu \left[\left(\int_0^t W_2^{\varepsilon,ij}(F_{\varepsilon^{-2}s}) ds \right)^2 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E}_\mu \left[\left(t \int_{[0,\tau]} \int_{B^c(0,\varepsilon^{-1}\delta)} y_i y_j \nu(z, dy) \pi(dz) \right)^2 \right] \right)^{\frac{1}{2}} \\ &\leq \frac{2C^{1/2} \varepsilon t^{1/2}}{c^{1/2}} \sup_{x \in [0,\tau]} \left| \int_{B(0,\varepsilon^{-1}\delta)} y_i y_j \left(\nu(x, dy) - \int_{[0,\tau]} \nu(z, dy) \pi(dz) \right) \right| \\ &\quad + t \left| \int_{[0,\tau]} \int_{B^c(0,\varepsilon^{-1}\delta)} y_i y_j \nu(z, dy) \pi(dz) \right| \\ &\leq \frac{4C^{1/2} \varepsilon t^{1/2}}{c^{1/2}} \sup_{x \in [0,\tau]} \int_{B(0,\varepsilon^{-1}\delta)} |y|^2 \nu(x, dy) + t \sup_{x \in [0,\tau]} \int_{B(0,\varepsilon^{-1}\delta)} |y|^2 \nu(x, dy) \quad (5.6)\end{aligned}$$

and

$$\begin{aligned}
& \left(\mathbb{E}_\mu \left[\left(\frac{1}{\varepsilon^2} \int_0^t \int_{B^c(0, \varepsilon^{-1}\delta)} h_i(\varepsilon y) h_j(\varepsilon y) \nu(F_{\varepsilon^2 s}, dy) ds \right)^2 \right] \right)^{\frac{1}{2}} \\
& \leq \left(\mathbb{E}_\mu \left[\left(\int_0^t W_3^{\varepsilon, ij}(F_{\varepsilon^{-2} s}) ds \right)^2 \right] \right)^{\frac{1}{2}} \\
& \quad + \left(\mathbb{E}_\mu \left[\left(\frac{t}{\varepsilon^2} \int_{[0, \tau]} \int_{B^c(0, \varepsilon^{-1}\delta)} h_i(\varepsilon y) h_j(\varepsilon y) \nu(z, dy) \pi(dz) \right)^2 \right] \right)^{\frac{1}{2}} \\
& \leq \frac{2C^{1/2} \|g\|_\infty t^{1/2}}{c^{1/2} \varepsilon} \sup_{x \in [0, \tau]} \left| \nu(x, B^c(0, \varepsilon^{-1}\delta)) - \int_{[0, \tau]} \nu(z, B^c(0, \varepsilon^{-1}\delta)) \pi(dz) \right| \\
& \quad + \frac{\|h\|_\infty t}{\varepsilon^2} \int_{[0, \tau]} \nu(z, B^c(0, \varepsilon^{-1}\delta)) \pi(dz) \\
& \leq \frac{4C^{1/2} \|g\|_\infty \varepsilon t^{1/2}}{c^{1/2} \delta^2} \sup_{x \in [0, \tau]} \int_{B^c(0, \varepsilon^{-1}\delta)} |y|^2 \nu(x, dy) + \frac{\|h\|_\infty t}{\delta^2} \int_{B^c(0, \varepsilon^{-1}\delta)} |y|^2 \nu(x, dy). \tag{5.7}
\end{aligned}$$

Finally, by combining (5.5), (5.6), (5.7), (C5) and (C6), we get (5.4). \square

In the Lévy process case we can generalize Theorem 3.1.

Corollary 5.2. *Let $\{F_t\}_{t \geq 0}$ be a d -dimensional Lévy process with Lévy triplet $(b, c, \nu(dy))$ satisfying condition (C5). Then, for any initial distribution of $\{F_t\}_{t \geq 0}$,*

$$\left\{ \varepsilon \left(F_{\varepsilon^{-2}t} - \varepsilon^{-2}t \left(b + \int_{B^c(0,1)} y \nu(dy) \right) \right) \right\}_{t \geq 0} \xrightarrow[\varepsilon \searrow 0]{d} \{W_t\}_{t \geq 0},$$

where $\{W_t\}_{t \geq 0}$ is a d -dimensional zero-drift Brownian motion starting from the origin and determined by a covariance matrix of the form

$$\Sigma := \left(c_{ij} + \int_{\mathbb{R}^d} y_i y_j \nu(dy) \right)_{1 \leq i, j \leq d}. \tag{5.8}$$

Proof. First, define

$$\bar{F}_t := F_t - t \left(b + \int_{B^c(0,1)} y \nu(dy) \right), \quad t \geq 0.$$

Clearly, $\{\bar{F}_t\}_{t \geq 0}$ is a Lévy process determined by a symbol of the form

$$\bar{q}(\xi) = q(\xi) + i \left\langle \xi, b + \int_{B^c(0,1)} y \nu(dy) \right\rangle,$$

where $q(\xi)$ is the symbol of $\{F_t\}_{t \geq 0}$. Next, as in the proof of Theorem 3.1, we conclude that for any initial distribution $\mu(dx)$ of $\{F_t\}_{t \geq 0}$, $\{\varepsilon \bar{F}_{\varepsilon^{-2}t}\}$, $\varepsilon > 0$, are \mathbb{P}_μ -semimartingales whose (modified)

characteristics (relative to a truncation function $h(x)$) are given by

$$\begin{aligned} B(h)_t^{\varepsilon,i} &= \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{R}^d} (h_i(\varepsilon y) - \varepsilon y_i 1_{B(0,1)}(y)) \nu(dy) ds - \frac{t}{\varepsilon} \int_{B^c(0,1)} y_i \nu(dy), \\ C_t^{\varepsilon,ij} &= t c_{ij}, \\ N^\varepsilon(ds, B) &= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} 1_B(\varepsilon y) \nu(dy) ds, \quad B \in \mathcal{B}(\mathbb{R}^d), \\ \tilde{C}(h)_t^{\varepsilon,ij} &= t c_{ij} + \frac{t}{\varepsilon^2} \int_{\mathbb{R}^d} h_i(\varepsilon y) h_j(\varepsilon y) \nu(dy) ds, \end{aligned}$$

$i, j = 1, \dots, d$. Again, according to [JS03, Theorem VIII.2.17], in order to prove the desired convergence it suffices to prove that

$$B(h)_t^{\varepsilon,i} \xrightarrow{\varepsilon \searrow 0} 0$$

for all $t \geq 0$ and all $i = 1, \dots, d$,

$$\int_0^t \int_{\mathbb{R}^d} g(y) N^\varepsilon(ds, dy) \xrightarrow{\varepsilon \searrow 0} 0$$

for all $t \geq 0$ and all $g \in C_b(\mathbb{R}^d)$ vanishing in a neighborhood of the origin, and

$$\tilde{C}(h)_t^{\varepsilon,ij} \xrightarrow{\varepsilon \searrow 0} \Sigma^{ij}$$

for all $t \geq 0$ and all $i, j = 1, \dots, d$, where Σ is given in (5.8). But, the above relations can be easily verified by employing (C5). \square

Now, we prove Theorem 3.7.

Proof of Theorem 3.7. Denote by $\mathbb{D}(\mathbb{R}^d)$ the space of all \mathbb{R}^d -valued càdlàg functions $\alpha : [0, \infty) \rightarrow \mathbb{R}^d$. It is well known that $\mathbb{D}(\mathbb{R}^d)$ admits a metrizable topology, called the Skorokhod topology, for which $\mathbb{D}(\mathbb{R}^d)$ is Polish space (a complete metrizable space). For more on the Skorokhod topology and space $\mathbb{D}(\mathbb{R}^d)$ we refer the readers to [Bil68] and [JS03]. Next, let $O \subseteq \mathbb{R}^d$ be an arbitrary set. Define a function $\tau_O : \mathbb{D}(\mathbb{R}^d) \rightarrow [0, \infty]$ by

$$\tau_O(\alpha) := \inf\{t > 0 : \alpha(t) \notin O\}.$$

If O is open, according to [Har77, Lemma 8], the set of continuity points of $\tau_O(\alpha)$ contains the set

$$\mathcal{C} := \left\{ \alpha \in \mathbb{D}(\mathbb{R}^d) : \lim_{r \rightarrow 0} \tau_{O^+}(\alpha) = \lim_{r \rightarrow 0} \tau_{O^-}(\alpha) = \tau_O(\alpha) \right\},$$

where

$$O^{r+} := \{x \in \mathbb{R}^d : d(x, O) < r\}, \quad O^{r-} := \{x \in \mathbb{R}^d : d(x, O^c) > r\}$$

and $d(x, y) := |x - y|$, $x, y \in \mathbb{R}^d$, denotes the standard Euclidian metric on \mathbb{R}^d . Assume that O satisfies the assumptions from the statement of the theorem and let us prove that \mathcal{C} contains the sample paths of $\{W_t\}_{t \geq 0}$ \mathbb{P}_x -a.s., $x \in \mathbb{R}^d$. Clearly, in order to prove the assertion, it suffices to prove that

$$\mathbb{P}_x(\tau_{O \cup \partial O}(\{W_t\}_{t \geq 0}) = 0) = \mathbb{P}_x(\tau_{O^c}(\{W_t\}_{t \geq 0}) = 0) = 1, \quad x \in \partial O. \quad (5.9)$$

If $d = 1$, the assertion immediately follows from Blumenthal's 0-1 law (see [MP10, Theorem 2.8]). Assume that $d \geq 2$. Then, according to [MP10, Theorem 8.3],

$$\mathbb{P}_x(\tau_O(\{W_t\}_{t \geq 0}) = 0) = \mathbb{P}_x(\tau_{\text{Int } O^c}(\{W_t\}_{t \geq 0}) = 0) = 1, \quad x \in \partial O.$$

Clearly,

$$\begin{aligned} & \{\omega \in \Omega : \tau_O(\{W_t(\omega)\}_{t \geq 0}) = 0\} \setminus \{\omega \in \Omega : \tau_{O \cup \partial O}(\{W_t(\omega)\}_{t \geq 0}) = 0\} \\ & \subseteq \{\omega \in \Omega : \text{there exists } n \geq 1, \text{ such that } W_t(\omega) \in \partial O \text{ for all } t \in [0, 1/n]\} \end{aligned}$$

and, similarly,

$$\begin{aligned} & \{\omega \in \Omega : \tau_{\text{Int } O^c}(\{W_t(\omega)\}_{t \geq 0}) = 0\} \setminus \{\omega \in \Omega : \tau_{O^c}(\{W_t(\omega)\}_{t \geq 0}) = 0\} \\ & \subseteq \{\omega \in \Omega : \text{there exists } n \geq 1, \text{ such that } W_t(\omega) \in \partial O \text{ for all } t \in [0, 1/n]\}. \end{aligned}$$

Furthermore, for any $x \in \mathbb{R}^d$,

$$\begin{aligned} & \mathbb{P}_x(\{\omega \in \Omega : \text{there exists } n \geq 1, \text{ such that } W_t(\omega) \in \partial O \text{ for all } t \in [0, 1/n]\}) \\ & = \mathbb{P}_x\left(\bigcup_{n \geq 1} \{\omega \in \Omega : W_t(\omega) \in \partial O \text{ for all } t \in [0, 1/n]\}\right) \\ & \leq \sum_{n \geq 1} \mathbb{P}_x(W_{1/n} \in \partial O) \\ & = 0, \end{aligned}$$

where in the final step we employed the assumption that ∂O has Lebesgue measure zero. Hence, the relation in (5.9) follows.

Now, we conclude that the set of continuity points of the function $\mathcal{T}_O : \mathbb{D}(\mathbb{R}^d) \rightarrow \mathbb{D}(\mathbb{R}^d) \times [0, \infty]$, $\mathcal{T}_O(\alpha) := (\alpha, \tau_O(\alpha))$, also contains the sample paths of $\{W_t\}_{t \geq 0}$ \mathbb{P}_x -a.s., $x \in \mathbb{R}^d$. Consequently, by Theorem 3.1 and [Bil68, Theorem 5.1], for any initial distribution of $\{F_t\}_{t \geq 0}$,

$$\left(\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0}, T^\varepsilon\right) = \mathcal{T}_O\left(\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0}\right) \xrightarrow[\varepsilon \searrow 0]{d} \mathcal{T}_O(\{W_t\}_{t \geq 0}) = (\{W_t\}_{t \geq 0}, T^0), \quad (5.10)$$

which together with [Ald81, Corollary 2.5] proves (3.5).

To prove the relation in (3.6), define a function $\mathcal{S}_O : \mathbb{D}(\mathbb{R}^d) \rightarrow \mathbb{D}(\mathbb{R}^d)$ by $\mathcal{S}_O(\alpha) := \alpha(\cdot \wedge \tau_O(\alpha))$, and let us prove that the set of continuity points of $\mathcal{S}_O(\alpha)$ also contains the sample paths of $\{W_t\}_{t \geq 0}$ \mathbb{P}_x -a.s., $x \in \mathbb{R}^d$. According to [JS03, Proposition VI.1.17], it suffices to show that for every $\omega \in \Omega$ such that $\{W_t(\omega)\}_{t \geq 0}$ is continuous and every sequence $\{\alpha_n\}_{n \geq 1} \subseteq \mathbb{D}(\mathbb{R}^d)$ converging (in the Skorokhod topology) to $\{W_t(\omega)\}_{t \geq 0}$, $\mathcal{S}_O(\alpha_n) \xrightarrow{n \nearrow \infty} \mathcal{S}_O(\{W_t(\omega)\}_{t \geq 0})$ locally uniformly (or, equivalently, in the Skorokhod topology). Fix an arbitrary $\omega \in \Omega$ such that $\{W_t(\omega)\}_{t \geq 0}$ is continuous. If $\tau_O(\{W_t(\omega)\}_{t \geq 0}) = \infty$, the claim easily follows from the continuity of the function $\tau_O(\alpha)$ at sample paths of $\{W_t\}_{t \geq 0}$ \mathbb{P}_x -a.s., $x \in \mathbb{R}^d$. Assume now that $\tau_O(\{W_t(\omega)\}_{t \geq 0}) = t < \infty$, and let $\{\alpha_n\}_{n \geq 1}$ be an arbitrary sequence in $\mathbb{D}(\mathbb{R}^d)$ converging to $\{W_t(\omega)\}_{t \geq 0}$ locally uniformly. Because of the continuity of $\{W_t(\omega)\}_{t \geq 0}$, for an arbitrary $\epsilon > 0$ there exist $0 < \delta_\epsilon < t$ and $n_\epsilon \geq 1$, such that $|W_s(\omega) - W_u(\omega)| < \epsilon$ for all $s, u \in [t - \delta_\epsilon, t + \delta_\epsilon]$, $|W_s(\omega) - \alpha_n(s)| < \epsilon$ for all $s \in [t - \delta_\epsilon, t + \delta_\epsilon]$ and all $n \geq n_\epsilon$ and $|\tau_O(\alpha_n) - t| < \delta_\epsilon$ for all $n \geq n_\epsilon$. If K is any compact subset of $[0, t - \delta_\epsilon]$ or $[t + \delta_\epsilon, \infty)$,

the claim follows trivially. On the other hand, if $K = [t - \epsilon, t + \epsilon]$,

$$\begin{aligned}
& \sup_{s \in K} |\mathcal{S}_O(\alpha_n) - \mathcal{S}_O(\{W_t(\omega)\}_{t \geq 0})| \\
&= \sup_{s \in K} |\alpha_n(s \wedge \tau_O(\alpha_n)) - W_{s \wedge t}(\omega)| \\
&\leq \sup_{s \in K} |\alpha_n(s \wedge \tau_O(\alpha_n)) - W_{s \wedge \tau_O(\alpha_n)}(\omega)| + \sup_{s \in K} |W_{s \wedge \tau_O(\alpha_n)}(\omega) - W_{s \wedge t}(\omega)| \\
&\leq \sup_{s \in K} |\alpha_n(s) - W_s(\omega)| + \sup_{s, u \in K} |W_s(\omega) - W_u(\omega)| \\
&\leq 2\epsilon.
\end{aligned}$$

Finally, similarly as in (5.10), Theorem 3.1 and [Bil68, Theorem 5.1] imply that for any initial distribution of $\{F_t\}_{t \geq 0}$,

$$\{\varepsilon F_{\varepsilon^{-2}(t \wedge T^\varepsilon)}\}_{t \geq 0} = \mathcal{S}_O\left(\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0}\right) \xrightarrow[\varepsilon \searrow 0]{d} \mathcal{S}_O(\{W_t\}_{t \geq 0}) = \{W_{t \wedge T^0}\}_{t \geq 0},$$

which is (3.6). \square

6 Non-Zero Drift Case

In this section, we discuss the non-zero drift case. As we commented in the first section, in [BLP78] and [Bha85] the authors have considered the homogenization of a d -dimensional diffusion $\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0}$ determined by τ -periodic coefficients $b(x) = (b_i(x))_{1 \leq i \leq d}$ and $c(x) = (c_{ij})_{1 \leq i, j \leq d}$, such that $c(x)$ is symmetric and uniformly elliptic, $b_i \in C^1(\mathbb{R}^d)$ with Hölderian derivative and $c_{ij} \in C^2(\mathbb{R}^d)$ with Hölderian first derivative (in particular, these assumptions ensure that $\{F_t\}_{t \geq 0}$ satisfies (C3) and (C4)). More precisely, in [BLP78, Lemma 3.4.1] they have first shown that there exist unique τ -periodic $\beta_i \in C^2(\mathbb{R}^d)$, $i = 1, \dots, d$, satisfying

$$\mathcal{A}^\infty \beta_i(x) = b_i(x) - \bar{b}_i, \quad x \in \mathbb{R}^d,$$

where \mathcal{A}^∞ denotes the Feller generator of $\{F_t\}_{t \geq 0}$, $\bar{b}_i := \int_{[0, \tau]} b_i(x) \pi(dx)$ and $\pi(dx)$ is again the unique invariant probability measure associated to $\{F_t^\tau\}_{t \geq 0}$. Then, by employing the above representation of the drift function, they have obtained that

$$\{\varepsilon F_{\varepsilon^{-2}t} - \varepsilon^{-1} \bar{b}t\}_{t \geq 0} \xrightarrow[\varepsilon \searrow 0]{d} \{\bar{W}_t\}_{t \geq 0}$$

for any initial distribution of $\{F_t\}_{t \geq 0}$. Here, $\bar{b} := (\bar{b}_i)_{1 \leq i \leq d}$ and $\{\bar{W}_t\}_{t \geq 0}$ is a d -dimensional zero-drift Brownian motion starting from the origin and determined by the covariance matrix given in (1.4).

On the other hand, if $\{F_t\}_{t \geq 0}$ is a diffusion with jumps with Lévy triplet $(b(x), c(x), \nu(x, dy))$ satisfying conditions (C3)-(C6). Then, by using the approach through the characteristics of semi-martingales, the convergence

$$\{\varepsilon F_{\varepsilon^{-2}t} - \varepsilon^{-1} \bar{b}t\}_{t \geq 0} \xrightarrow[\varepsilon \searrow 0]{d} \{\tilde{W}_t\}_{t \geq 0}$$

reduces to the convergences

$$\frac{1}{\varepsilon} \int_0^t (b_i(F_{\varepsilon^{-2}s}) - \bar{b}_i) ds \xrightarrow[\varepsilon \searrow 0]{\mathbb{P}_\mu} 0, \quad i = 1, \dots, d, \quad (6.1)$$

(5.2), (5.3) and (5.4), where $\mu(dx)$ is an arbitrary initial distribution of $\{F_t\}_{t \geq 0}$ and $\{\tilde{W}_t\}_{t \geq 0}$ is a d -dimensional Brownian motion starting from the origin whose covariance matrix has to be determined. Note that $\{F_t - \bar{b}t\}_{t \geq 0}$ is again a diffusion with jumps which satisfies (C3)-(C6). However, the relation in (6.1) never holds, unless $b(x) = \bar{b}$ (dx -a.e.). Let us be more precise. Let $\{P_t^\tau\}_{t \geq 0}$ be the semigroup of $\{F_t^\tau\}_{t \geq 0}$ (with respect to $(B_b([0, \tau]), \|\cdot\|_\infty)$). Observe that, since $\pi(dx)$ is an invariant probability measure for $\{F_t^\tau\}_{t \geq 0}$, $\{P_t^\tau\}_{t \geq 0}$ can be naturally (and uniquely) extended to a positivity preserving contraction semigroup on $(L^2([0, \tau], \pi(dx)), \|\cdot\|_2)$, which we again denote by $\{P_t^\tau\}_{t \geq 0}$. Indeed, for an arbitrary $f \in L^2([0, \tau], \pi(dx))$, we have

$$\|f\|_2^2 = \int_{[0, \tau]} f^2(x) \pi(dx) = \int_{[0, \tau]} P_t^\tau f^2(x) \pi(dx) \geq \int_{[0, \tau]} (P_t^\tau f(x))^2 \pi(dx) = \|P_t^\tau f\|_2^2.$$

The positivity of $\{P_t^\tau\}_{t \geq 0}$ is trivially satisfied. Next, the infinitesimal generator $(\mathcal{A}_\tau^2, \mathcal{D}_{\mathcal{A}_\tau^2})$ of the semigroup $\{P_t^\tau\}_{t \geq 0}$, with respect to $(L^2([0, \tau], \pi(dx)), \|\cdot\|_2)$, is a linear operator $\mathcal{A}_\tau^2 : \mathcal{D}_{\mathcal{A}_\tau^2} \rightarrow L^2([0, \tau], \pi(dx))$ defined by

$$\mathcal{A}_\tau^2 f := \lim_{t \rightarrow 0} \frac{P_t^\tau f - f}{t}, \quad f \in \mathcal{D}_{\mathcal{A}_\tau^2} := \left\{ f \in L^2([0, \tau], \pi(dx)) : \lim_{t \rightarrow 0} \frac{P_t^\tau f - f}{t} \text{ exists in } \|\cdot\|_2 \right\}.$$

Now, [Bha82, Theorem 2.1] states that if $f = \mathcal{A}_\tau^2 g$ for some $g \in \mathcal{D}_{\mathcal{A}_\tau^2}$ (note that in this case, due to the stationarity of $\pi(dx)$, $\int_{[0, \tau]} f(x) \pi(dx) = 0$), then for any initial distribution of $\{F_t\}_{t \geq 0}$,

$$\left\{ \varepsilon^{-1} \int_0^t f(F_{\varepsilon^{-2}s}) ds \right\}_{t \geq 0} \xrightarrow[\varepsilon \searrow 0]{d} \{W_t^1\}_{t \geq 0},$$

where $\{W_t^1\}_{t \geq 0}$ is a one-dimensional zero-drift Brownian motion starting from the origin and determined by the variance parameter $\sigma^2 = -2 \int_{[0, \tau]} f(x) g(x) \pi(dx)$. In what follows we show that every τ -periodic $f \in B_b(\mathbb{R}^d)$, that is, its restriction to $[0, \tau]$, satisfying $\int_{[0, \tau]} f(x) \pi(dx) = 0$ is always contained in the range of \mathcal{A}_τ^2 (see also [Bha82, Remark 2.3.1]). Thus, due to the fact that $p(t, x, dy)$, $t > 0$, $x \in \mathbb{R}^d$, and dy are mutually absolutely continuous, [Bha82, Proposition 2.4] implies that (6.1) holds if, and only if, $b(x) = \bar{b}$ (dx -a.e.).

Proposition 6.1. *Let $\{F_t\}_{t \geq 0}$ be a d -dimensional diffusion with jumps with B_b -generator $(\mathcal{A}^b, \mathcal{D}_{\mathcal{A}^b})$, satisfying conditions (C3) and (C4). Then,*

$$\mathcal{E} := \{f \in C^2(\mathbb{R}^d) : f(x) \text{ is } \tau\text{-periodic}\} \subseteq \mathcal{D}_{\mathcal{A}^b}, \quad \mathcal{E}_\tau := \{f|_{[0, \tau]} : f \in \mathcal{E}\} \subseteq \mathcal{D}_{\mathcal{A}_\tau^2}$$

and, on \mathcal{E}_τ (that is, \mathcal{E}), $\mathcal{A}_\tau^2 f|_{[0, \tau]} = (\mathcal{A}^b f)|_{[0, \tau]}$.

Proof. First, we show that $\mathcal{E} \subseteq \mathcal{D}_{\mathcal{A}^b}$ and, on this class of functions, \mathcal{A}^b has the representation (1.1). Let $\{P_t\}_{t \geq 0}$ be the semigroup of $\{F_t\}_{t \geq 0}$ and let $\mathcal{L} : C_b^2(\mathbb{R}^d) \rightarrow B_b(\mathbb{R}^d)$ be defined by the relation in (1.1), where $C_b^k(\mathbb{R}^d)$, $k \geq 0$, denotes the space of k times differentiable functions such that all derivatives up to order k are bounded. Observe that actually $\mathcal{L} : C_b^2(\mathbb{R}^d) \rightarrow C_b(\mathbb{R}^d)$ (see [Sch98a, Remark 4.5]). Further, by [Sch98b, Corollary 3.6],

$$\mathbb{E}_x \left[f(F_t) - \int_0^t \mathcal{L} f(F_s) ds \right] = f(x), \quad x \in \mathbb{R}^d, \quad f \in C_b^2(\mathbb{R}^d).$$

Consequently, for any τ -periodic $f \in C^2(\mathbb{R}^d)$,

$$\begin{aligned} \lim_{t \rightarrow 0} \left\| \frac{P_t f - f}{t} - \mathcal{L}f \right\|_\infty &= \lim_{t \rightarrow 0} \left\| \frac{1}{t} \int_0^t (P_s \mathcal{L}f - \mathcal{L}f) ds \right\|_\infty \\ &\leq \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t \sup_{x \in [0, \tau]} |P_s \mathcal{L}f(x) - \mathcal{L}f(x)| ds \\ &= 0, \end{aligned}$$

where in the second step we used the fact that $x \mapsto \mathcal{L}f(x)$ is also τ -periodic and in the final step we applied [Jac01, Lemma 4.8.7].

Finally, we show that $\mathcal{E}_\tau \subseteq \mathcal{D}_{\mathcal{A}_\tau^2}$ and $\mathcal{A}_\tau^2 f|_{[0, \tau]} = (\mathcal{A}^b f)|_{[0, \tau]}$, $f \in \mathcal{E}$. Let $f \in \mathcal{E}$. Then, by using the τ -periodicity and (4.2),

$$\begin{aligned} \lim_{t \rightarrow 0} \left\| \frac{P_t^\tau f|_{[0, \tau]} - f|_{[0, \tau]}}{t} - (\mathcal{A}^b f)|_{[0, \tau]} \right\|_2 &\leq \lim_{t \rightarrow 0} \left\| \frac{P_t^\tau f|_{[0, \tau]} - f|_{[0, \tau]}}{t} - (\mathcal{A}^b f)|_{[0, \tau]} \right\|_\infty \\ &= \lim_{t \rightarrow 0} \left\| \frac{P_t f - f}{t} - \mathcal{A}^b f \right\|_\infty \\ &= 0, \end{aligned}$$

which concludes the proof. \square

Proposition 6.2. *Let $\{F_t\}_{t \geq 0}$ be a d -dimensional diffusion with jumps with semigroup $\{P_t\}_{t \geq 0}$, satisfying conditions (C3) and (C4). Then, $\{P_t^\tau\}_{t \geq 0}$ is strongly continuous with respect to $(L^2([0, \tau], \pi(dx)), \|\cdot\|_2)$, that is, $\lim_{t \rightarrow \infty} \|P_t^\tau f - f\|_2 = 0$ for all $f \in L^2([0, \tau], \pi(dx))$.*

Proof. First, recall that the space $C([0, \tau])$ is dense in $L^2([0, \tau], \pi(dx))$ with respect to $\|\cdot\|_2$ (see [Fol84, Proposition 7.9]). Next, according to the Stone-Weierstrass theorem (see [Fol84, Theorem 4.45]), $C([0, \tau])$ is the closure of the set \mathcal{E}_τ (with respect to $\|\cdot\|_\infty$). Therefore, \mathcal{E}_τ is dense in $L^2([0, \tau], \pi(dx))$ (with respect to $\|\cdot\|_2$). Further, because of the τ -periodicity and (4.2), it is easy to see that $\{P_t^\tau\}_{t \geq 0}$ is strongly continuous on \mathcal{E}_τ (even with respect to $\|\cdot\|_\infty$). Finally, let $f \in L^2([0, \tau], \pi(dx))$ and $\varepsilon > 0$ be arbitrary and let $f_\varepsilon \in \mathcal{E}_\tau$ be such that $\|f - f_\varepsilon\|_2 < \varepsilon$. Then,

$$\begin{aligned} \|P_t^\tau f - f\|_2 &\leq \|P_t^\tau f - P_t^\tau f_\varepsilon\|_2 + \|P_t^\tau f_\varepsilon - f_\varepsilon\|_2 + \|f_\varepsilon - f\|_2 \\ &\leq 2\|f - f_\varepsilon\|_2 + \|P_t^\tau f_\varepsilon - f_\varepsilon\|_2 \\ &\leq 2\varepsilon + \|P_t^\tau f_\varepsilon - f_\varepsilon\|_2, \end{aligned}$$

which proves the assertion. \square

Proposition 6.3. *Let $\{F_t\}_{t \geq 0}$ be a d -dimensional diffusion with jumps with semigroup $\{P_t\}_{t \geq 0}$, satisfying conditions (C3) and (C4). Then, for any τ -periodic $f \in B_b(\mathbb{R}^d)$ satisfying $\int_{[0, \tau]} f(x) \pi(dx) = 0$ there exists $g \in \mathcal{D}_{\mathcal{A}_\tau^2}$ such that $f(x) = \mathcal{A}_\tau^2 g(x)$.*

Proof. First, let $h_1, h_2 \in L^2([0, \tau], \pi(dx))$ be arbitrary functions satisfying $\int_{[0, \tau]} h_i(x) \pi(dx) = 0$, $i = 1, 2$. Then, since $\pi(dx)$ is an invariant probability measure for $\{F_t^\tau\}_{t \geq 0}$, we conclude

$$\int_{[0, \tau]} P_t^\tau h_1(x) h_2(x) \pi(dx) = \mathbb{E}_\pi^\tau [\mathbb{E}_\pi^\tau [h_1(F_{2t}^\tau) | F_t^\tau] h_2(F_t^\tau)] = \mathbb{E}_\pi^\tau [h_1(F_{2t}^\tau) h_2(F_t^\tau)], \quad t \geq 0.$$

Now, by employing this fact, the proof of [EK86, Corollary 7.2.7] and (4.5) imply

$$\begin{aligned}
& \left| \int_{[0,\tau]} P_t^\tau h_1(x) h_2(x) \pi(dx) \right| \\
& \leq 2 \left(\sup \{ |\mathbb{E}_\pi^\tau [1_B | F_t^\tau] - \pi(B)| : \omega \in \Omega, B \in \sigma\{F_u^\tau : u \geq 2t\}\} \right)^{1/2} \|h_1\|_2 \|h_2\|_2 \\
& \leq 2 \left(\sup \{ |P_t^\tau 1_B(x) - \pi(B)| : x \in [0, \tau], B \in \mathcal{B}([0, \tau]) \} \right)^{1/2} \|h_1\|_2 \|h_2\|_2 \\
& \leq 2C^{1/2} \|h_1\|_2 \|h_2\|_2 e^{-ct/2}.
\end{aligned}$$

Specially, for any $s, t \geq 0$,

$$\|P_t^\tau h_1\|_2^2 = \left| \int_{[0,\tau]} (P_t^\tau h_1(x))^2 \pi(dx) \right| \leq 2C^{1/2} \|h_1\|_2 \|P_t^\tau h_1\|_2 e^{-ct/2} \leq 2C^{1/2} \|h_1\|_2^2 e^{-ct/2}$$

and

$$\left| \int_{[0,\tau]} P_t^\tau h_1(x) P_s^\tau h_2(x) \pi(dx) \right| \leq 2C^{1/2} \|h_1\|_2 \|P_s^\tau h_2\|_2 e^{-ct/2} \leq 2^{3/2} C^{3/4} \|h_1\|_2 \|h_2\|_2 e^{-ct/2 - cs/4}.$$

Next, observe that for each $t \geq 0$,

$$\int_0^t P_s^\tau f ds \in L^2([0, \tau], \pi(dx)).$$

Now, we prove that $\lim_{t \rightarrow \infty} \int_0^t P_s^\tau f ds$ exists in $L^2([0, \tau], \pi(dx))$. Let $\{t_n\}_{n \geq 1}$ be an arbitrary sequence in $[0, \infty)$, $t_n \nearrow \infty$. Then, the sequence $\left\{ \int_0^{t_n} P_s^\tau f ds \right\}_{n \geq 1}$ is a Cauchy sequence in $L^2([0, \tau], \pi(dx))$. Indeed, for arbitrary $n, m \in \mathbb{N}$, $n \leq m$,

$$\begin{aligned}
\left\| \int_0^{t_m} P_s^\tau f ds - \int_0^{t_n} P_s^\tau f ds \right\|_2^2 &= \int_{[0,\tau]} \left(\int_{t_n}^{t_m} P_s^\tau f(x) ds \right)^2 \pi(dx) \\
&= \int_{[0,\tau]} \int_{t_n}^{t_m} \int_{t_n}^{t_m} P_t^\tau f(x) P_s^\tau f(x) ds dt \pi(dx) \\
&\leq \int_{t_n}^{t_m} \int_{t_n}^{t_m} \left| \int_{[0,\tau]} P_t^\tau f(x) P_s^\tau f(x) \pi(dx) \right| ds dt \\
&\leq 2^{3/2} C^{3/4} \|f\|_2^2 \int_{t_n}^{t_m} \int_{t_n}^{t_m} e^{-ct/2 - cs/4} ds dt.
\end{aligned}$$

Thus, since $L^2([0, \tau], \pi(dx))$ is a Hilbert space, the sequence $\left\{ \int_0^{t_n} P_s^\tau f ds \right\}_{n \geq 1}$ converges to some $g \in L^2([0, \tau], \pi(dx))$. By completely the same reasoning as above, it is easy to see that the function $g(x)$ does not depend on the choice of a sequence $\{t_n\}_{n \geq 1}$. Therefore,

$$\lim_{t \rightarrow \infty} \int_0^t P_s^\tau f ds = g$$

in $L^2([0, \tau], \pi(dx))$. Furthermore, because of the continuity (boundedness) of $\{P_t^\tau\}_{t \geq 0}$, for any $u \geq 0$ we have

$$P_u^\tau g = \lim_{t \rightarrow \infty} P_u^\tau \left(\int_0^t P_s^\tau f ds \right) = \lim_{t \rightarrow \infty} \int_u^t P_s^\tau f ds. \quad (6.2)$$

Finally, we show that $g \in \mathcal{D}_{\mathcal{A}_\tau^2}$ and $\mathcal{A}_\tau^2(-g) = f$. We have

$$\lim_{t \rightarrow 0} \left(\frac{P_t^\tau(-g) + g}{t} - f \right) = \lim_{t \rightarrow 0} \left(\frac{\int_0^t P_s^\tau f ds}{t} - f \right) = \lim_{t \rightarrow 0} \frac{\int_0^t (P_s^\tau f - f) ds}{t} = 0,$$

where in the first step we used (6.2) and in the final step we employed the strong continuity property of $\{P_t^\tau\}_{t \geq 0}$ (Proposition 6.2). \square

7 Long-Time Behavior of Periodic Diffusions with Small Jumps: Transience, Recurrence and Ergodicity

In this section, as one of the applications of Theorem 3.1, we discuss transience, recurrence and ergodicity of periodic diffusions with small jumps. Denote by $\text{Leb}(dx)$ the Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$. Recall that a càdlàg strong Markov process $(\Omega, \mathcal{F}, \{\mathbb{P}_x\}_{x \in \mathbb{R}^d}, \{\mathcal{F}_t\}_{t \geq 0}, \{\theta_t\}_{t \geq 0}, \{M_t\}_{t \geq 0})$, denoted by $\{M_t\}_{t \geq 0}$ in the sequel, is called

- (i) *Lebesgue-irreducible* if $\text{Leb}(B) > 0$ implies $\int_0^\infty \mathbb{P}_x(M_t \in B) dt > 0$ for all $x \in \mathbb{R}^d$;
- (ii) *recurrent* if it is Lebesgue-irreducible and if $\text{Leb}(B) > 0$ implies $\int_0^\infty \mathbb{P}_x(M_t \in B) dt = \infty$ for all $x \in \mathbb{R}^d$;
- (iii) *Harris recurrent* if it is Lebesgue-irreducible and if $\text{Leb}(B) > 0$ implies $\mathbb{P}_x(\tau_B < \infty) = 1$ for all $x \in \mathbb{R}^d$, where $\tau_B := \inf\{t \geq 0 : M_t \in B\}$;
- (iv) *transient* if it is Lebesgue-irreducible and if there exists a countable covering of \mathbb{R}^d with sets $\{B_j\}_{j \in \mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^d)$, such that for each $j \geq 1$ there is a finite constant $c_j \geq 0$ such that $\int_0^\infty \mathbb{P}_x(M_t \in B_j) dt \leq c_j$ holds for all $x \in \mathbb{R}^d$.

It is well known that every Lebesgue-irreducible Markov process is either transient or recurrent (see [Twe94, Theorem 2.3]). Also, clearly, every Harris recurrent Markov process is recurrent. But, in general, these two properties are not equivalent. They differ on a set of the Lebesgue measure zero (see [Twe94, Theorem 2.5]). However, for a diffusion with jumps satisfying condition (C3) these two properties are indeed equivalent (see [San14a, Proposition 2.1]). Obviously, a diffusion with jumps satisfying the conditions in (C3) is always Lebesgue-irreducible. Now, we recall, for our purposes more adequate, characterization of the transience and recurrence through the sample-paths of the underlying process. Let $B \in \mathcal{B}(\mathbb{R}^d)$ be arbitrary. The *sets of transient and recurrent functions* in $\mathbb{D}(\mathbb{R}^d)$, with respect to B , are defined as

$$\begin{aligned} T(B) &:= \{\alpha \in \mathbb{D}(\mathbb{R}^d) : \exists s \geq 0 \text{ such that } \alpha(t) \notin B, \forall t \geq s\} \\ R(B) &:= \{\alpha \in \mathbb{D}(\mathbb{R}^d) : \forall n \in \mathbb{N}, \exists t \geq n \text{ such that } \alpha(t) \in B\}, \end{aligned}$$

respectively. Recall that $\mathbb{D}(\mathbb{R}^d)$ denotes the space of \mathbb{R}^d -valued càdlàg functions endowed with the Skorokhod topology. It is clear that $T(B) = R(B)^c$ and for any open set $O \subseteq \mathbb{R}^d$, due to the right continuity of the elements of $\mathbb{D}(\mathbb{R}^d)$, $T(O)$ and $R(O)$ are measurable (with respect to the Borel σ -algebra generated by the Skorokhod topology). Assume now that $\{M_t\}_{t \geq 0}$ is a Lebesgue-irreducible Markov process and that for every compact set $K \subseteq \mathbb{R}^d$ there exists $t_K > 0$ such that

$$\inf_{x \in K} \mathbb{P}_x(M_{t_K} \in B) > 0 \tag{7.1}$$

holds for all $B \in \mathcal{B}(\mathbb{R}^d)$ satisfying $\text{Leb}(B) > 0$. Then, [San14b, Proposition 2.4] asserts that $\{M_t\}_{t \geq 0}$ is transient if, and only if,

$$\mathbb{P}_x(\{M_t\}_{t \geq 0} \in T(O)) = 1$$

for all $x \in \mathbb{R}^d$ and all open bounded sets $O \subseteq \mathbb{R}^d$, and it is recurrent if, and only if,

$$\mathbb{P}_x(\{M_t\}_{t \geq 0} \in R(O)) = 1$$

for all $x \in \mathbb{R}^d$ and all open bounded sets $O \subseteq \mathbb{R}^d$. According to [Sch98a, Corollary 3.4] and [SW12, Corollary 2.2], any diffusion with jumps satisfying **(C3)** automatically satisfies the relation in (7.1).

Theorem 7.1. *Let $\{F_t\}_{t \geq 0}$ be a d -dimensional diffusion with jumps satisfying conditions **(C3)**-**(C6)** and let $\{W_t\}_{t \geq 0}$ be a d -dimensional zero-drift Brownian motion determined by the relation in (3.1). Then, $\{F_t\}_{t \geq 0}$ is transient (recurrent) if, and only if, $\{W_t\}_{t \geq 0}$ is transient (recurrent).*

Proof. As we commented above, it suffices to prove that $\mathbb{P}_x(\{F_t\}_{t \geq 0} \in T(O)) = 1$ for all $x \in \mathbb{R}^d$ and all open bounded sets $O \subseteq \mathbb{R}^d$ if, and only if, $\mathbb{P}_x(\{W_t\}_{t \geq 0} \in T(O)) = 1$ for all $x \in \mathbb{R}^d$ and all open bounded sets $O \subseteq \mathbb{R}^d$, and $\mathbb{P}_x(\{F_t\}_{t \geq 0} \in R(O)) = 1$ for all $x \in \mathbb{R}^d$ and all open bounded sets $O \subseteq \mathbb{R}^d$ if, and only if, $\mathbb{P}_x(\{W_t\}_{t \geq 0} \in R(O)) = 1$ for all $x \in \mathbb{R}^d$ and all open bounded sets $O \subseteq \mathbb{R}^d$. To see this, first, by [Fra06, Lemma 3], we have $\mathbb{P}_x(\{W_t\}_{t \geq 0} \in \partial T(O)) = \mathbb{P}_x(\{W_t\}_{t \geq 0} \in \partial R(O)) = 0$ for all $x \in \mathbb{R}^d$ and all open bounded sets $O \subseteq \mathbb{R}^d$. Consequently, due to Theorem 3.1 and [Bil68, Theorem 2.1], for any initial distribution $\mu(dx)$ of $\{F_t\}_{t \geq 0}$ and any open bounded set $O \subseteq \mathbb{R}^d$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}_\mu(\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0} \in T(O)) = \mathbb{P}_0(\{W_t\}_{t \geq 0} \in T(O))$$

and

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}_\mu(\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0} \in R(O)) = \mathbb{P}_0(\{W_t\}_{t \geq 0} \in R(O)).$$

Next, since the processes $\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0}$, $\varepsilon > 0$, also satisfy **(C3)**, $\mathbb{P}_\mu(\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0} \in T(O))$ and $\mathbb{P}_\mu(\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0} \in R(O))$ are either 0 or 1 for all initial distributions $\mu(dx)$ of $\{F_t\}_{t \geq 0}$, all $\varepsilon > 0$ and all open bounded sets $O \subseteq \mathbb{R}^d$. Thus, in order to prove the desired result, it suffices to prove that the functions

$$\varepsilon \mapsto \mathbb{P}_\mu(\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0} \in T(O)) \quad \text{and} \quad \varepsilon \mapsto \mathbb{P}_\mu(\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0} \in R(O))$$

are continuous for all initial distributions $\mu(dx)$ of $\{F_t\}_{t \geq 0}$ and all open bounded sets $O \subseteq \mathbb{R}^d$. But this fact follows directly from

$$\begin{aligned} \{\omega \in \Omega : \{\varepsilon F_{\varepsilon^{-2}t}(\omega)\}_{t \geq 0} \in T(O)\} &= \{\omega \in \Omega : \{F_t(\omega)\}_{t \geq 0} \in T(\varepsilon^{-1}O)\}, \\ \{\omega \in \Omega : \{\varepsilon F_{\varepsilon^{-2}t}(\omega)\}_{t \geq 0} \in R(O)\} &= \{\omega \in \Omega : \{F_t(\omega)\}_{t \geq 0} \in R(\varepsilon^{-1}O)\} \end{aligned}$$

and the characterization of the transience and recurrence through the sample paths of $\{F_t\}_{t \geq 0}$. Here, for $a \in \mathbb{R}$ and $B \subseteq \mathbb{R}^d$, $aB := \{ab : b \in B\}$. \square

At the end, let us comment the (strong) ergodicity of periodic diffusions with small jumps. First, recall that if $\{M_t\}_{t \geq 0}$ is a recurrent Markov process, then it admits a unique (up to constant multiples) invariant measure (see [Twe94, Theorem 2.6]). If the invariant measure is finite, then it may be normalized to a probability measure. If $\{M_t\}_{t \geq 0}$ is recurrent with finite invariant measure, then $\{M_t\}_{t \geq 0}$ is called *positive recurrent*, otherwise it is called *null recurrent*. One would expect that every positive recurrent process is (strongly) ergodic, but in general this is not true (see [MT93])

and [PS15]). In the case of a Lebesgue-irreducible diffusion with jumps $\{F_t\}_{t \geq 0}$, these properties are equivalent. Indeed, according to [MT93, Theorem 6.1] and [SW13, Theorem 3.3] it suffices to show that if $\{F_t\}_{t \geq 0}$ admits an invariant probability measure $\pi(dx)$, then it is recurrent. Assume that $\{F_t\}_{t \geq 0}$ is transient. Then there exists a countable covering of \mathbb{R}^d with sets $\{B_j\}_{j \in \mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^d)$, such that for each $j \geq 1$ there is a finite constant $c_j \geq 0$ such that $\int_0^\infty \mathbb{P}_x(F_t \in B_j) dt \leq c_j$ holds for all $x \in \mathbb{R}^d$. Let $t > 0$ be arbitrary. Then for each $j \geq 1$ we have

$$t\pi(B_j) = \int_0^t \int_{\mathbb{R}^d} \mathbb{P}_x(F_s \in B_j) \pi(dx) ds \leq c_j.$$

In particular, by letting $t \rightarrow \infty$ we deduce that $\pi(B_j) = 0$ for all $j \geq 1$, which is impossible.

Now, directly from Theorem 7.1, we conclude that a d -dimensional, $d \geq 3$, diffusion with jumps $\{F_t\}_{t \geq 0}$ satisfying (C3)-(C6) is never (strongly) ergodic. Recall that a d -dimensional zero-drift Brownian motion is recurrent if, and only if, $d = 1, 2$ (see [Sat99, Corollary 37.6]). Moreover, $\{F_t\}_{t \geq 0}$ is neither (strongly) ergodic in $d = 1, 2$. Indeed, if this were the case, then $\{F_t\}_{t \geq 0}$ would possess an invariant probability measure, say $\pi(dx)$. Consequently, for any $\varepsilon > 0$, $\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0}$ also possesses invariant probability measure which is given by $\pi^\varepsilon(dx) := \pi(\varepsilon^{-1}dx)$. To see this, first note that $\{\varepsilon F_{\varepsilon^{-2}t}\}_{t \geq 0}$ is a Markov processes with respect to $\mathbb{P}_x^\varepsilon(d\omega) := \mathbb{P}_{\varepsilon^{-1}x}(d\omega)$, $x \in \mathbb{R}^d$. Thus, for all $\varepsilon > 0$, all $t \geq 0$ and all $B \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \mathbb{P}_x^\varepsilon(\varepsilon F_{\varepsilon^{-2}t} \in B) \pi^\varepsilon(dx) = \int_{\mathbb{R}^d} \mathbb{P}_{\varepsilon^{-1}x}(\varepsilon F_{\varepsilon^{-2}t} \in B) \pi(\varepsilon^{-1}dx) = \pi(\varepsilon^{-1}B) = \pi^\varepsilon(B).$$

Finally, by fixing an arbitrary $t_0 > 0$, Theorem 3.1 implies that

$$\begin{aligned} \mathbb{P}_0(W_{t_0} \in B(0, 1)) &= \lim_{\varepsilon \rightarrow 0} \mathbb{P}_\pi(\varepsilon F_{\varepsilon^{-2}t_0} \in B(0, 1)) \\ &= \lim_{\varepsilon \rightarrow 0} \mathbb{P}_{\pi^\varepsilon}^\varepsilon(\varepsilon F_{\varepsilon^{-2}t_0} \in B(0, 1)) \\ &= \lim_{\varepsilon \rightarrow 0} \pi^\varepsilon(B(0, 1)) \\ &= 1, \end{aligned}$$

which leads to a contradiction.

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